Confidential

NEW/OLD

Department of Examination - Sri Lanka

G.C.E. (A/L) Examination - 2020

10 - Combined Mathematics - II

NEW/OLD Syllabus

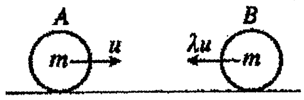
Marking Scheme

This document has been prepared for the use of Marking Examiners. Some changes would be made according to the views presented at the Chief Examiners' meeting.

Amendments to be included

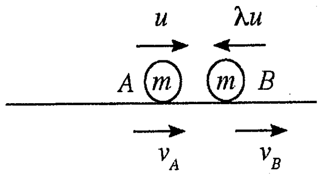
New Syllabus

1. Two particles A and B each of mass m, moving in the same straight line on a smooth horizontal floor, but in opposite directions collide directly. The velocities of A and B just before collision are u and λu, respectively. The coefficient of restitution between A and B is $\frac{1}{2}$.



Find the velocity of A just after collision and show that if $\lambda > \frac{1}{3}$, then the direction of motion of A is reversed.

For A and B, applying $\underline{I} = \Delta(m\underline{v})$, $\rightarrow$:



$(mv_A + mv_B) - (mu - m\lambda u) = 0$

$v_A + v_B = (1 - \lambda)u$ ——— (1) **(10)**

(5) if only one moment um is correct

Newton's Expermental law :

$v_B - v_A = \frac{1}{2}(u + \lambda u)$ ——— (2) **(5)**

(1) - (2) : $2v_A = u - \lambda u - \frac{1}{2}u - \frac{\lambda}{2}u$

$v_A = \frac{1}{4}(1 - 3\lambda)u$ **(5)**

If $\lambda > \frac{1}{3}$, then $v_A < 0$. **(5)**

$\therefore$ The direction of motion of A is reversed.

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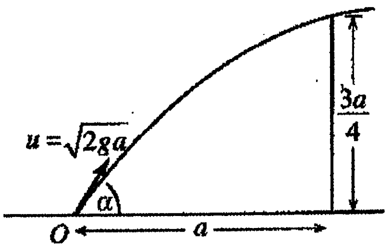
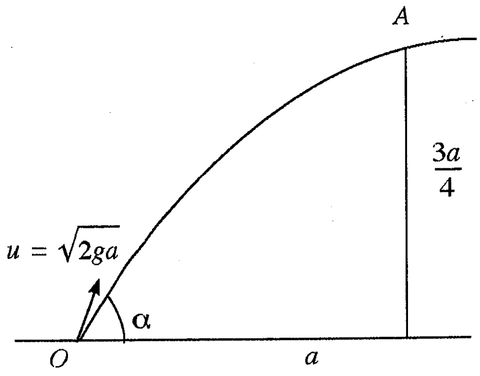
2. A particle is projected from a point O on a horizontal floor with initial velocity $u = \sqrt{2ga}$ and at an angle $\alpha \left(0 < \alpha < \frac{\pi}{2}\right)$ to the horizontal. The particle just clears a vertical wall of height $\frac{3a}{4}$ located at a horizontal distance a from O.

Show that $\sec^2\alpha - 4\tan\alpha + 3 = 0$.

Hence, show that $\alpha = \tan^{-1}(2)$.





Let t be the time taken from O to A

Applying $S = ut + \frac{1}{2}at^2$:

$\rightarrow$ $a = u\cos\alpha\, t$ —— (1) (5)

$\uparrow$ $\frac{3a}{4} = u\sin\alpha\, t - \frac{1}{2}gt^2$ —— (2) (5)

(1) $\Rightarrow$ $t = \dfrac{a}{u\cos\alpha}$

Now (2) $\Rightarrow$ $\dfrac{3a}{4} = a\tan\alpha - \dfrac{1}{2}g\dfrac{a^2}{2ga\cos^2\alpha}$

$\Rightarrow \dfrac{3}{4} = \tan\alpha - \dfrac{1}{4}\sec^2\alpha$

$\Rightarrow \sec^2\alpha - 4\tan\alpha + 3 = 0$ (5)

$\Rightarrow (1 + \sec^2\alpha) - 4\tan\alpha + 3 = 0$ (5)

$\tan^2\alpha - 4\tan\alpha + 4 = 0$

$\Rightarrow (\tan\alpha - 2)^2 = 0$

$\tan\alpha = 2$

$\therefore\ \alpha = \tan^{-1}(2)$. (5)

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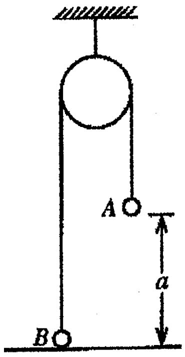
3. Two particles A and B, each of mass m, attached to the two ends of a light inextensible string which passes over a fixed smooth pulley are in equilibrium with the particle A at a height a from a horizontal floor and the particle B touching the floor, as shown in the figure. Now, the particle A is given an impulse mu vertically downwards. Find the velocity of the particle A just after the impulse. Write down the time taken by A to reach the floor.

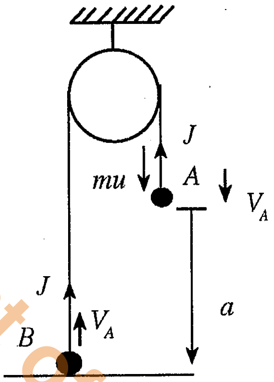


Applying $\underline{I} = \Delta(m\underline{v})$

Ⓐ ↓ $mu - J = mV_A$ ⑤

Ⓑ ↑ $J = mV_A$ ⑤

$\therefore \; V_A = \dfrac{u}{2}$ ⑤



⑤

$T = \dfrac{a}{V_A} = \dfrac{2a}{u}$ ⑤

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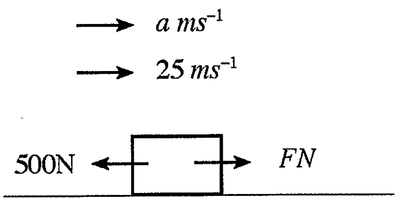
4. A car of mass 1500 kg travels on a straight horizontal road against a constant resistance of magnitude 500 N. Find the acceleration of the car when the engine of the car is working at power 50 kW and the car is travelling with speed 25 m s^{-1}.
At this instant, the engine of the car is turned off. Find the speed of the car after 50 seconds from the instant the engine was turned off.



Since the power = 50kW, we have

$$50 \times 10^3 = F \times 25 \quad (5)$$

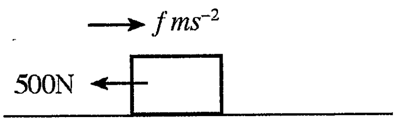
$$\Rightarrow F = 2000$$

Applying $\underline{F} = m\underline{a} \longrightarrow$

$$F - 500 = 1500\,a \quad (5)$$

$$a = 1 \quad (5)$$

When the engine of the car is turned off,



$\underline{F} = m\underline{a} \longrightarrow$

$$-500 = 1500\,f \quad (5)$$

$$f = -\frac{1}{3}$$

Applying $v = u + at \longrightarrow$ $v = 25 - \frac{1}{3} \times 50$

$$v = \frac{25}{3}\ ms^{-1} \quad (5)$$

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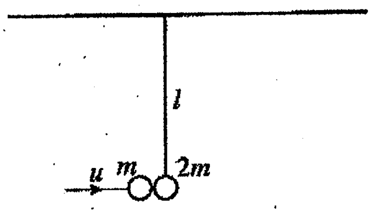
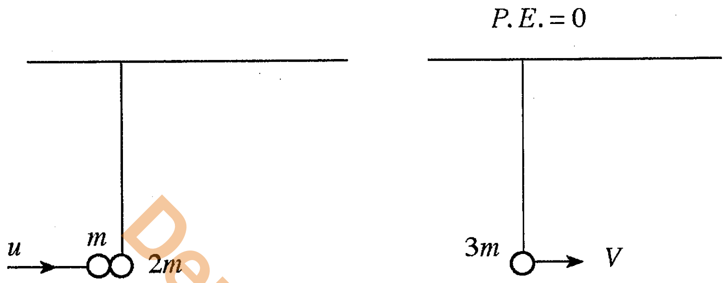
5. A particle P of mass $2m$, hanging freely from a horizontal ceiling by a light inextensible string of length l, is in equilibrium. Another particle of mass m moving in a horizontal direction with velocity u collides with the particle P and coalesces to it. The string remains taut after the collision and the composite particle just reaches the ceiling. Show that $u = \sqrt{18gl}$.





Applying $\underline{I} = \Delta (m\underline{v})$

for m and $2m \longrightarrow 0 = 3mV - (mu)$ (5)

$V = \frac{u}{3}$ (5)

Applying the principle of conservation of energy for the composite particle:

$\frac{1}{2}(3m)V^2 - 3mgl = 0$ (10)

$V^2 = 2gl$

$\frac{u^2}{9} = 2gl$

$u = \sqrt{18gl}$ (5)

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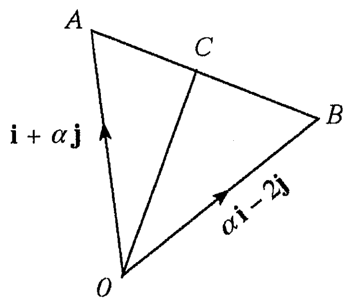
6. Let $\alpha > 0$ and in the usual notation, let $\mathbf{i}+\alpha\mathbf{j}$ and $\alpha\mathbf{i}-2\mathbf{j}$ be the position vectors of two points A and B, respectively, with respect to a fixed origin O. Also, let C be the point on AB such that $AC:CB = 1:2$. It is given that OC is perpendicular to AB. Find the value of α.



$$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$$

$$= -(\mathbf{i}+\alpha\mathbf{j}) + (\alpha\mathbf{j} - z\mathbf{j}) \quad (5)$$

$$= (\alpha - 1)\,\mathbf{i} - (\alpha + 2)\,\mathbf{j}$$

$$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC}$$

$$= \overrightarrow{OA} + \frac{1}{3}\overrightarrow{AB} \quad (5)$$

$$= (\mathbf{i}+\alpha\mathbf{j}) + \frac{1}{3}[(\alpha-1)\,\mathbf{i} - (\alpha+2)\,\mathbf{j}] \quad (5)$$

$$= (\mathbf{i}+\alpha\mathbf{j}) + \frac{1}{3}[(\alpha-1)\,\mathbf{i} - (\alpha+1)\,\mathbf{j}]$$

$$= \frac{1}{3}[(\alpha+2)\,\mathbf{i} + 2(\alpha-1)\,\mathbf{j}]$$

$$\overrightarrow{OC} \perp \overrightarrow{AB} \Leftrightarrow \overrightarrow{OC} \cdot \overrightarrow{AB} = 0 \quad (5)$$

$$\Leftrightarrow (\alpha-1)(\alpha+2) - 2(\alpha+2)(\alpha-1) = 0$$

$$\Leftrightarrow (\alpha-1)(\alpha+2) = 0$$

$$\Leftrightarrow \alpha = 1 \quad (5) \qquad (\because \alpha > 0)$$

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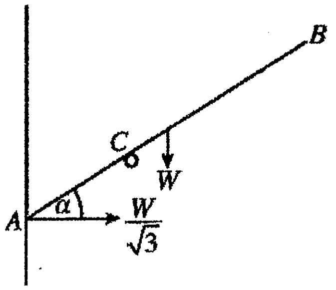
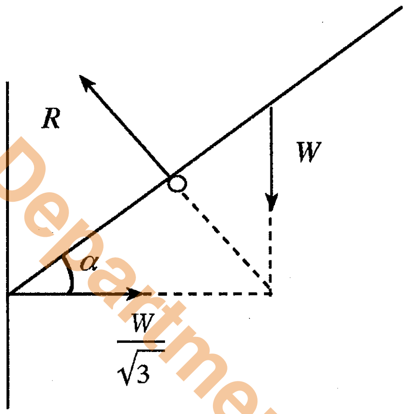
7. A uniform rod ACB of length $2a$ and weight W is kept in equilibrium with the end A against a smooth vertical wall by a smooth peg placed at C, as shown in the figure. It is given that the reaction at A from the wall is $\frac{W}{\sqrt{3}}$. Show that the angle α that the rod makes with the horizontal is $\frac{\pi}{6}$.

Show also that $AC = \frac{3}{4}a$.





For the equilibrium of the rod:

$\rightarrow$ $R \sin\alpha = \frac{W}{\sqrt{3}}$ ——— (1) (5)

$\uparrow$ $R\cos\alpha = W$ ——— (2) (5)

$\frac{(1)}{(2)} \Rightarrow \tan\alpha = \frac{1}{\sqrt{3}}$

$\Rightarrow \alpha = \frac{\pi}{6}$ (5)

Now (1) $\Rightarrow R = \frac{2W}{\sqrt{3}}$

$A\curvearrowleft$ $R \times AC = W \times a\cos\frac{\pi}{6}$ (or $Wa\cos\alpha$) (5)

$\frac{2W}{\sqrt{3}} \times AC = W \times a \times \frac{\sqrt{3}}{2}$

$AC = \frac{3}{4}a$ (5)

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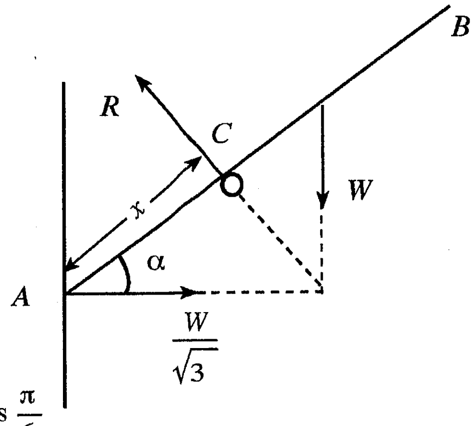
Aliter 1



AB ↗ $\quad \frac{W}{\sqrt{3}} \cos\alpha = W \sin\alpha$ (10)

$\Rightarrow \quad \tan\alpha = \frac{1}{\sqrt{3}}$

$\Rightarrow \quad \alpha = \frac{\pi}{6}$ (5)

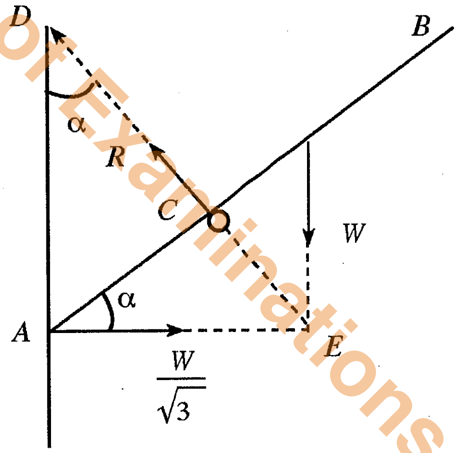
C ↶ $\quad \frac{W}{\sqrt{3}} \times x \sin\frac{\pi}{6} = W \times (a - x)\cos\frac{\pi}{6}$ (5)

$\frac{1}{\sqrt{3}} \times x \times \frac{1}{2} = (a - x)\,\frac{\sqrt{3}}{2}$

$x = 3\,(a - x)$

$x = \frac{3}{4}\,a$ (5)

Aliter 2



$\Delta\ ADE$ is a force Δ (5)

$$\frac{\frac{W}{\sqrt{3}}}{AE} = \frac{W}{AD}$$

$$\frac{AE}{AD} = \frac{1}{\sqrt{3}} \quad (5)$$

$$\Rightarrow \tan\alpha = \frac{1}{\sqrt{3}}$$

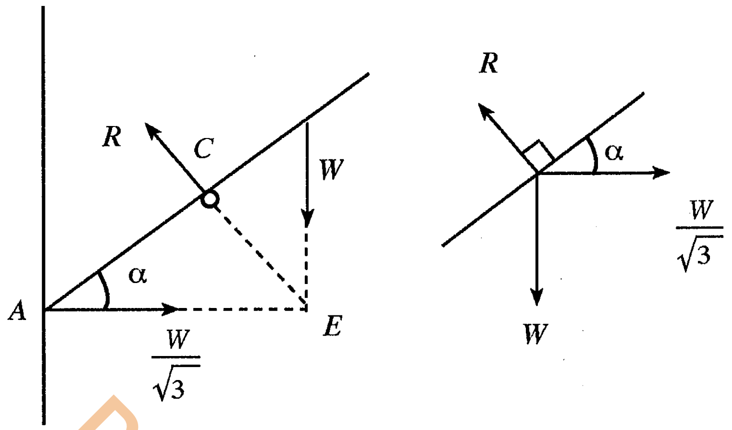
$$\Rightarrow \alpha = \frac{\pi}{6} \quad (5)$$

$$\therefore AE = a\cos\frac{\pi}{6} = \frac{a\sqrt{3}}{2} \quad (5)$$

$$AC = AE\cos\frac{\pi}{6} = \frac{a\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2}$$

$$= \frac{3}{4}\,a \quad (5)$$

Aliter 3



By Lami's Rule:

$$\frac{W}{\sin\left(\frac{\pi}{2} + \alpha\right)} = \frac{\frac{W}{\sqrt{3}}}{\sin(\pi - \alpha)} \quad (5)$$

$$\frac{1}{\cos\alpha} = \frac{1}{\sqrt{3}\sin\alpha} \quad (5)$$

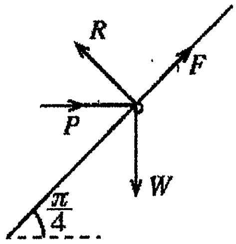
$$\Rightarrow \tan\alpha = \frac{1}{\sqrt{3}}$$

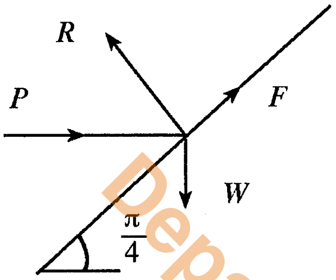
$$\Rightarrow \alpha = \frac{\pi}{6} \quad (5)$$

$\overset{\frown}{C}$ ***OR*** from Δ ACE we get $AC = \frac{3}{4}a$. (5) + (5)

8. A small bead of weight W is threaded to a fixed rough straight wire inclined at an angle $\frac{\pi}{4}$ to the horizontal. The bead is kept in equilibrium by a horizontal force of magnitude P as shown in the figure. The coefficient of friction between the bead and the wire is $\frac{1}{2}$.
Obtain equations sufficient to determine the frictional force F and the normal reaction R on the bead, in terms of P and W.

It is given that $\frac{F}{R}=\frac{W-P}{W+P}$. Show that $\frac{W}{3} \leq P \leq 3W$.





$$F = \frac{W-P}{W+P}$$

For the equilibrium of the bead:

 $F - \frac{W}{\sqrt{2}} + \frac{P}{\sqrt{2}} = 0.$ (5) (or with $\cos\frac{\pi}{4}$, $\sin\frac{\pi}{4}$)

 $R - \frac{W}{\sqrt{2}} - \frac{P}{\sqrt{2}} = 0.$ (5) (or with $\cos\frac{\pi}{4}$, $\sin\frac{\pi}{4}$)

OR 

$$\mu \geq \frac{|F|}{R}$$

$$\frac{1}{2} \geq \frac{|W-P|}{W+P} \quad (10)$$

Only (5) without the absolute value

$$|W-P| \leq \frac{1}{2}(W+P)$$

$$-\frac{1}{2}(W+P) \leq W-P \leq \frac{1}{2}(W+P)$$

$$-W-P \leq 2W-2P \leq W+P$$

$$\frac{W}{3} \leq P \leq 3W \quad (5)$$

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9. Let A and B be two events of a sample space Ω. In the usual notation, it is given that $P(A)=\frac{3}{5}$, $P(B|A)=\frac{1}{4}$ and $P(A\cup B)=\frac{4}{5}$. Find $P(B)$.
Show that the events A and B are not independent.

$$P(B|A) = \frac{P(B\cap A)}{P(A)}$$

$$\Rightarrow \quad P(A\cap B) = \frac{3}{5}\times\frac{1}{4} = \frac{3}{20} \quad (5)$$

$$\Rightarrow \quad P(A\cup B) = P(A) + P(B) - P(A\cap B) \quad (5)$$

$$\frac{4}{5} = \frac{3}{5} + P(B) - \frac{3}{20}$$

$$P(B) = \frac{16}{20} - \frac{12}{20} + \frac{3}{20} = \frac{7}{20} \quad (5)$$

$$P(A)\,.\,P(B) = \frac{3}{5}\times\frac{7}{20} = \frac{21}{100} \quad (5)$$

$$\therefore \quad P(A\cap B) \neq P(A)\,.\,P(B) \quad (5)$$

$\therefore$ A and B are not independent.

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10. **A set of 5 observations of positive integers, each less than or equal to 10, has mean, media and mode each equals to 6. The range of the observations is 9. Find these five observations.**

Mode $= 6 \Rightarrow$ At least two of the numbers must be 6, 6 (5)

Range $= 9$ and the numbers are positive intergers ≤ 10, we have the smallest is 1 and the largest is 10. (5)

Since the median is 6, the numbers

must be $1, a, 6, 6, 10$ or $1, 6, 6, a, 10$. (5)

Mean $= 6$ gives $\frac{a+23}{5} = 6$. (5)

$\therefore \; a = 7$ (5)

$\therefore$ The numbers are 1, 6, 6, 7, 10.

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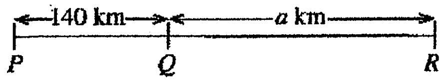
11. (*a*) Three railway stations P, Q and R located in a straight line such that $PQ = 140$ km and $QR = a$ km, as shown in the figure. At time $t = 0$, a train A starts from rest



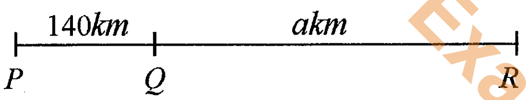
at P and moves towards Q with constant acceleration f km h^{-2} for half an hour and maintains the velocity it had at time $t = \frac{1}{2}$ h for three hours. Then it moves with constant retardation f km h^{-2} and comes to rest at Q. At time $t = 1$ h, another train B starts from rest at R and moves towards Q with constant acceleration $2f$ km h^{-2} for T hours and then with a constant retardation f km h^{-2} and comes to rest at Q. Both trains come to rest at the same instant. Sketch velocity-time graphs for the motions of A and B in the same diagram.
Hence or otherwise, show that $f = 80$ and find the values of T and a. **20**

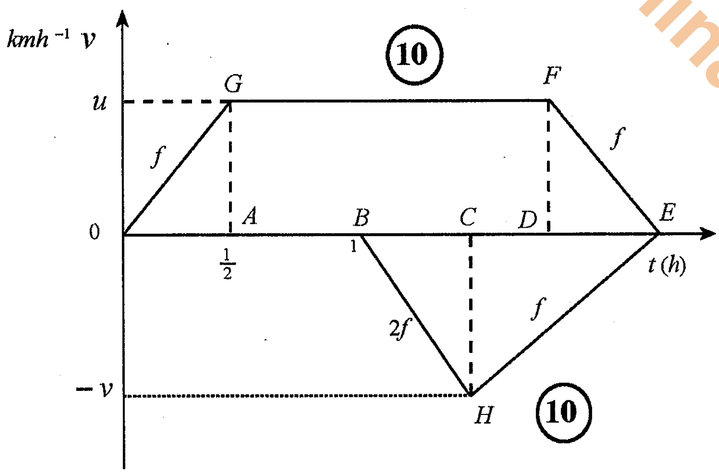
(*b*) A ship is sailing due west with uniform speed u relative to earth and a boat is sailing in a straight line path with uniform speed $\frac{u}{2}$ relative to earth. At a certain instant, the ship is at a distance d at an angle $\frac{\pi}{3}$ east of north from the boat.

(i) If the boat is sailing relative to earth in the direction making an angle $\frac{\pi}{6}$ west of north, show that the boat can intercept the ship and that the time taken by the boat to intercept the ship is $\frac{2d}{\sqrt{3}u}$.

(ii) If the boat is sailing relative to earth in the direction making an angle $\frac{\pi}{6}$ east of north, show that the speed of the boat relative to the ship is $\frac{\sqrt{7}u}{2}$ and that the shortest distance between the ship and the boat is $\frac{d}{2\sqrt{7}}$.

(*a*)





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$\underline{\Delta\, OAG}$

$f = \dfrac{u}{\frac{1}{2}}$

$\therefore \; f = 2u$

$\Delta\, OAG \equiv \Delta\, DEF$

$\therefore DE = \frac{1}{2}$ (5)

Area of the trapezium $OEFG = 140$ (5)

$\frac{1}{2}(4+3)\,u = 140$ (5)

$\therefore \; u = 40$

$\therefore \; f = 80.$ (5)

25

$\underline{\Delta\, BHC}$

$2f = \dfrac{V}{T} \Rightarrow 160 = \dfrac{V}{T}$ (5)

$\underline{\Delta\, ECH}$

$f = \dfrac{V}{CE} \Rightarrow 80 = \dfrac{V}{CE}$ (5)

$\therefore CE = 2T$ (5)

$\therefore 3T = 3$ and $T = 1.$ (5) Also $V = 160.$

a = Area of $\Delta\, BHE = \frac{1}{2} \times 3 \times 160$

$= 240$ (5)

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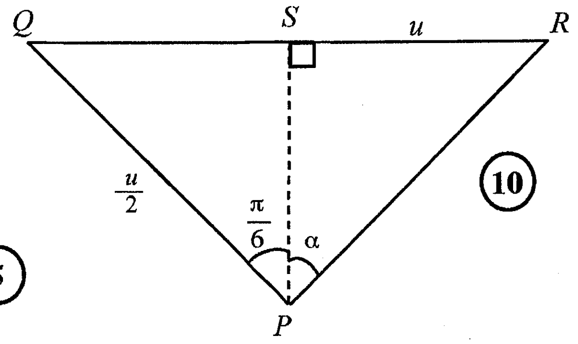
(*b*) $\mathbf{V}(S, E) = \leftarrow u$ (5)



(i) $\mathbf{V}(B, E) = \frac{u}{2}$ at $\frac{\pi}{6}$ to the vertical (5)

$\mathbf{V}(B, S) = \mathbf{V}(B, E) + \mathbf{V}(E, S)$ (5)

$= \overrightarrow{PQ} + \overrightarrow{QR}$

$= \overrightarrow{PR}$

$QS = \frac{u}{2} \sin \frac{\pi}{6} = \frac{u}{4}$

$\therefore\ SR = \frac{3u}{4}$

$SP = \frac{u}{2} \cos \frac{\pi}{6} = \frac{\sqrt{3}u}{4}$

$\tan\alpha = \frac{SR}{SP} = \frac{3u}{4} \times \frac{4}{\sqrt{3}u} = \sqrt{3}$ (10)

$\therefore\ \alpha = \frac{\pi}{3}$ (5)

$\therefore$ Boat can intercept the ship.

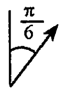
40

$Q\hat{P}R = \frac{\pi}{2}$

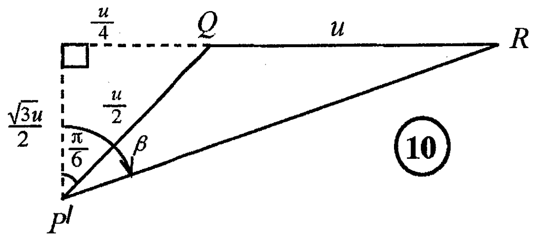
$\therefore\ PR = \frac{\sqrt{3}\,u}{2}$ (5)

$t = \frac{d}{PR} = \frac{2d}{\sqrt{3}\,u}$ (5)

10

(ii) $\mathbf{V}(B, E) =$ 

(5)

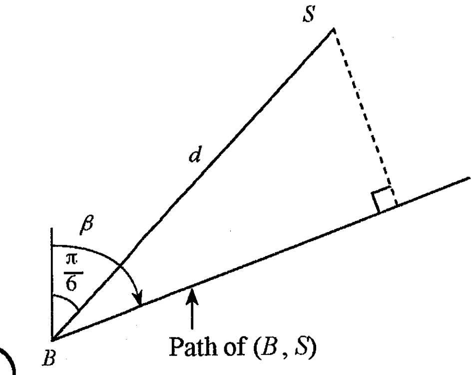


$\mathbf{V}(B, S) = \mathbf{V}(B, E) + \mathbf{V}(E, S)$

$= \overrightarrow{P'Q} + \overrightarrow{QR}$

$= \overrightarrow{P'R}$

From the velocity triangle,



$\sin\beta = \frac{5}{2\sqrt{7}} \quad \cos\beta = \frac{\sqrt{3}}{2\sqrt{7}}$

Shortest distance $= d\sin\left(\beta - \frac{\pi}{3}\right)$ (5)

$= d\left(\sin\beta\cos\frac{\pi}{3} - \cos\beta\sin\frac{\pi}{3}\right)$ (5)

$= d\left(\frac{5}{4\sqrt{7}} - \frac{3}{4\sqrt{7}}\right)$

$= \frac{d}{2\sqrt{7}}$ (5)

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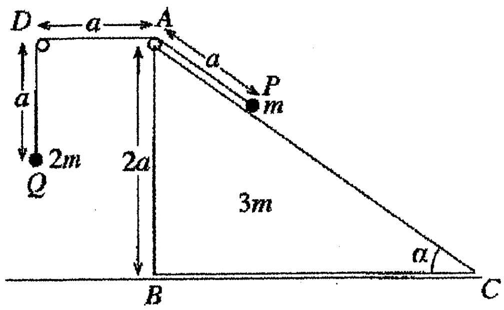
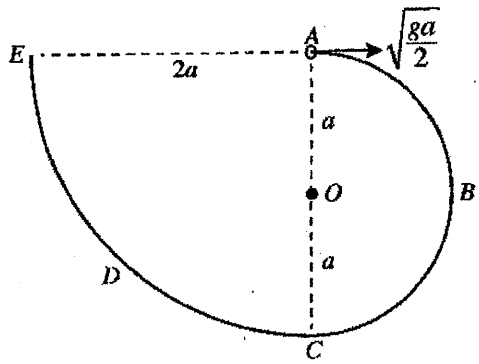
12.(*a*) The triangle ABC in the figure is the vertical cross-section through the centre of gravity of a smooth uniform wedge of mass $3m$ with $A\hat{C}B = \alpha$, $A\hat{B}C = \frac{\pi}{2}$ and $AB = 2a$ such that the face containing BC is placed on a smooth horizontal floor. The line AC is a line of greatest slope of the face containing it. The point D is a fixed point in the plane of ABC such that AD is horizontal. Two particles P and Q of masses m and $2m$, respectively are attached to the two ends of a light inextensible string of length $3a$ passing over smooth small pulleys fixed at A and D. The system is released from rest with the particle P held on AC and the particle Q hanging freely such that $AP = AD = DQ = a$, as shown in the figure. Obtain equations sufficient to determine the time taken by the particle Q to reach the floor.



(*b*) A smooth thin wire $ABCDE$ is fixed in a vertical plane, as shown in the figure. The portion ABC is a semicircle with centre O and radius a, and the portion CDE is a quarter of a circle with centre A and radius $2a$. The points A and C lie on the vertical line through O and the line AE is horizontal. A small smooth bead P of mass m is placed at A and is given a velocity $\sqrt{\frac{ga}{2}}$ horizontally, and begins to move along the wire.

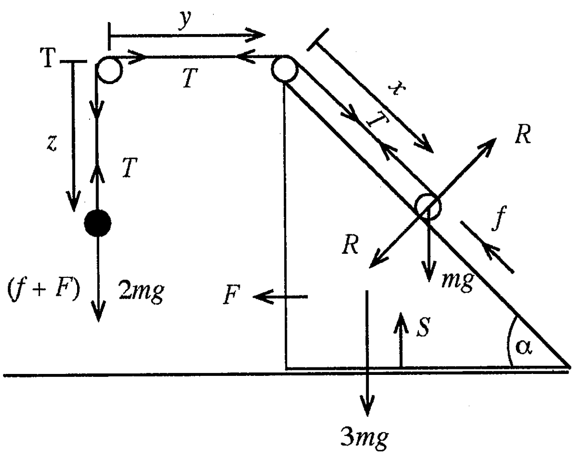


Show that the speed v of the bead P when $\overrightarrow{OP}$ makes an angle θ $(0 \le \theta \le \pi)$ with $\overrightarrow{OA}$ is given by $v^2 = \frac{ga}{2}(5 - 4\cos\theta)$.

Find the reaction on the bead P from the wire at the above position and show that it changes its direction when the bead P passes the point $\theta = \cos^{-1}\left(\frac{5}{6}\right)$.

Write down the velocity of the bead P just before it leaves the wire at E and find the reaction on the bead P from the wire at that instant.

(*a*)



Forces (15)

Accelerations (20)

$$\ddot{z} = -\ddot{x} - \ddot{y}$$
$$= f + F$$

Applying $F = ma$

For $(2m)$ $\downarrow$ $2mg - T = 2m(f + F)$ (10)

For (m) $\nwarrow$ $T - mg\sin\alpha = m(f + F\cos\alpha)$ (10)

For (m) and $(3m)$ $\leftarrow$ $T = 3mF + m(F + f\cos\alpha)$ (15)

$$a = \frac{1}{2}(f + F)t^2$$ (10)

80

(b)

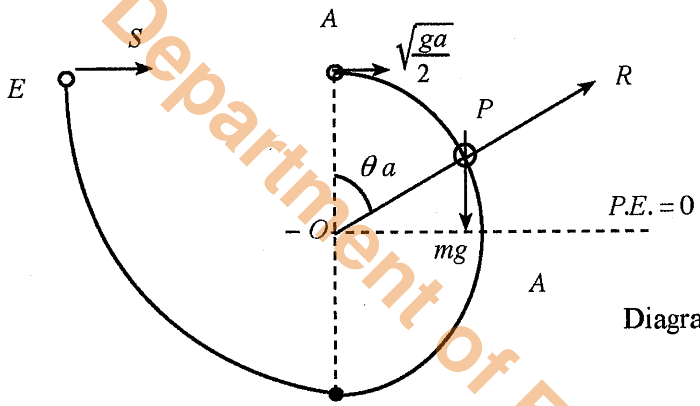


Diagram (10)

By the conservation of Energy,

P.E. + (5) K.E. + (5) equation (5)

$$\frac{1}{2}mv^2 + mga\cos\theta = \frac{1}{2}m\left(\frac{ga}{2}\right) + mga$$

$$2v^2 + 4ga\cos\theta = 5ga$$

$$v^2 = \frac{ga}{2}(5 - 4\cos\theta)$$ (5)

30

For circular motion, applying $\underline{F} = m\underline{a}$ $\nearrow$

$$R - mg\cos\theta = -m\frac{V^2}{a}$$ (10)

$$R = mg\cos\theta - \frac{mg}{2}(5 - 4\cos\theta)$$ (5)

$$= \frac{mg}{2}(6\cos\theta - 5)$$

$0 < \theta < \alpha$; $R > 0$ where $\cos\alpha = \frac{5}{6}$ (5)

$\alpha < \theta < \pi$; $R < 0$

Hence the reaction changes its direction when bead passes the point $\theta = \cos^{-1}\left(\frac{5}{6}\right)$.

20

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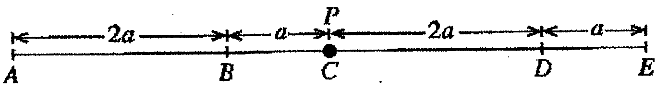
13. The points A, B, C, D and E lie on a straight line in that order, on a smooth horizontal table such that $AB = 2a$, $BC = a$, $CD = 2a$ and $DE = a$, as shown in the figure. One end of a light elastic string of natural length $2a$ and modulus of elasticity kmg is attached to the point A and the other end to a particle P of mass m. One end of another light elastic string of natural length a and modulus of elasticity mg is attached to the point E and the other end to the particle P. When the particle P is held at C and released, it stays in equilibrium. Find the value of k.



Now, the string AP is pulled until the particle P reaches the point D and released from rest. Show that the equation of motion of P from D to B is given by $\ddot{x}+\frac{3g}{a}x=0$, where $CP = x$.

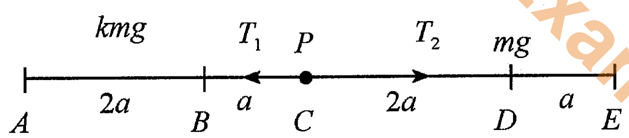
Using the formula $\dot{x}^2=\frac{3g}{a}(c^2-x^2)$, where c is the amplitude, show that the velocity of particle P when it reaches B is $3\sqrt{ga}$.

An impulse is given to the particle P when it reaches B so that the velocity of P just after the impulse is $\sqrt{ag}$ in the direction of $\overrightarrow{BA}$.

Show that the equation of motion of P after passing B until it comes to instantaneous rest is given by $\ddot{y}+\frac{g}{a}y=0$, where $DP = y$.

Show that the total time taken by the particle P, started at D, to reach B for the second time is $2\sqrt{\frac{a}{g}}\left(\frac{\pi}{3\sqrt{3}}+\cos^{-1}\left(\frac{3}{\sqrt{10}}\right)\right)$.

13.



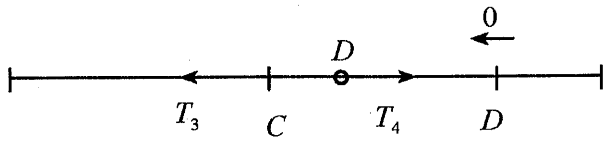
P is at equilibrium at C.

$\therefore\ T_1 - T_2 = 0$ (5)

$\Leftrightarrow kmg\,.\,\frac{a}{2a} = mg\,.\,\frac{2a}{a}$ (10)

$\Leftrightarrow k = 4$ (5)

20



For Ⓟ $\longrightarrow \underline{F} = m\underline{a}$

$$\therefore -T_3 + T_4 = m\ddot{x} \quad (5)$$

$$-4mg.\frac{(a+x)}{2a} + mg.\frac{(2a-x)}{a} = m\ddot{x} \quad (15)$$

$$\frac{g}{a}\left\{-2a - 2x + 2a - x\right\} = \ddot{x}$$

$$\ddot{x} = \frac{-3g}{a}x \quad (5)$$

$$\therefore \ddot{x} + \frac{3g}{a}x = 0$$

This is valid for $-a \leq x \leq 2a$

25

The centre for this S.H.M. is C and $\dot{x} = 0$ when $x = 2a$.

(5)

$\therefore$ Amplitude of this S.H.M. is $2a$. (5)

$$\therefore \dot{x}^2 = \frac{3g}{a}(4a^2 - x^2) \quad (5)$$

Let v be the speed at B $(x = -a)$.

$$\text{Then } v^2 = \frac{3g}{a}(4a^2 - a^2) \quad (5)$$

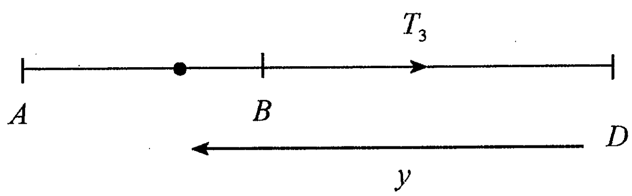
$$= 9ga$$

$$v = 3\sqrt{ga}$$

$\therefore$ velocity when P reaches B for the first time is $3\sqrt{ga} \longleftarrow$ (5)

25

Due to the impulse, velocity just after impulse is $\sqrt{ga}$.



$$-T_3 = m\ddot{y} \quad (5)$$

$$-mg\,\frac{y}{a} = m\ddot{y} \quad (5)$$

$$\therefore \ddot{y} = -\frac{g}{a}y$$

$$\text{or} \quad \ddot{y} + \frac{g}{a}y = 0 \quad (5)$$

15

The centre of this S.H.M. is D. (5)

Let c be the amplitude.

$$\dot{y} = \frac{g}{a}(c^2 - y^2)$$

$$\dot{y} = \sqrt{ga} \quad \text{when } y = 3a \quad (5)$$

$$ga = \frac{g}{a}(c^2 - 9a^2) \quad (5)$$

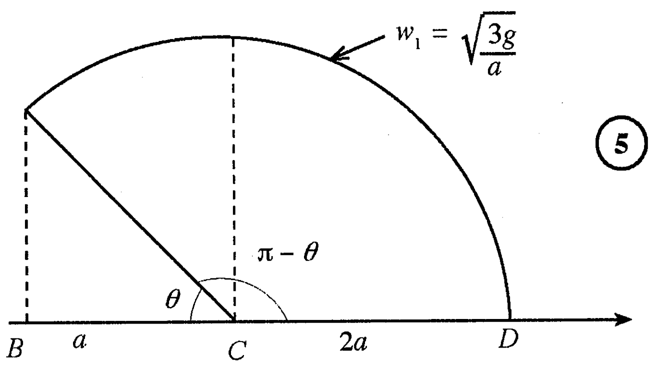
$$c^2 = 10a^2$$

$$c = \sqrt{10}\,a \quad (5)$$

Since $3a < \underbrace{\sqrt{10}\,a}_{c} < 5a$, the particle P will come to instantaneous rest at a point F between B and A.

20

Let τ_1 = Time taken from D to B.



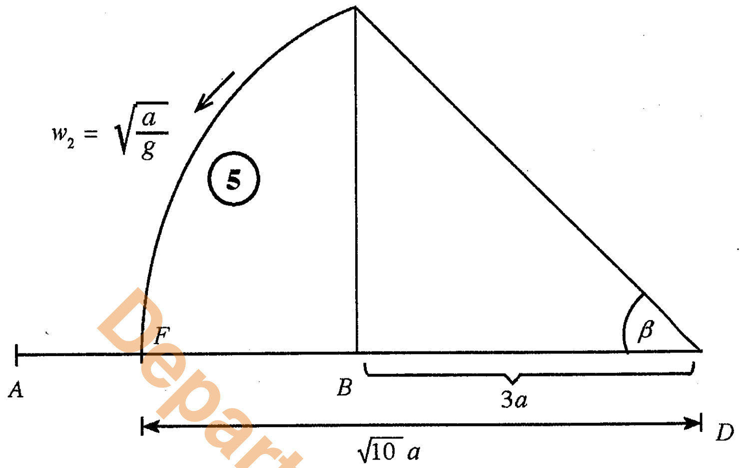
(5)

$$\sqrt{\frac{3g}{a}}\,\tau_1 = \pi - \theta, \quad (5) \qquad \text{where } \cos\theta = \frac{a}{2a}$$

$$\theta = \frac{\pi}{3} \quad (5)$$

$$\tau_1 = \sqrt{\frac{g}{3g}} \times \frac{2\pi}{3}$$

$$= \frac{2\pi}{3\sqrt{3}}\sqrt{\frac{a}{g}} \quad (5)$$



Let τ_2 = Time taken from B to F.

$\sqrt{\frac{a}{g}}\,\tau_2 = \beta$ (5) $\qquad \cos\beta = \frac{3a}{\sqrt{10}\,a}$

$\therefore \tau_2 = \sqrt{\frac{a}{g}}\cos^{-1}\left(\frac{3}{\sqrt{10}}\right)$ (5) $\qquad \beta = \cos^{-1}\left(\frac{3}{\sqrt{10}}\right)$

Let τ_3 = Time taken from F to B (Coming to B for the 2nd time)

$\tau_3 = \tau_2$

$\therefore$ The required time $= \tau_1 + 2\tau_2$ (5)

$$= 2\sqrt{\frac{a}{g}}\left\{\frac{\pi}{3\sqrt{3}} + \cos^{-1}\left(\frac{3}{\sqrt{10}}\right)\right\} \quad (5)$$

45

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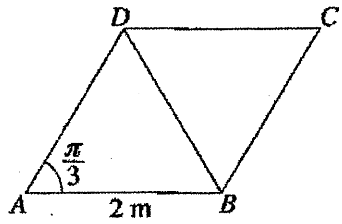
14. (*a*) Let **a** and **b** be two **unit vectors.**
The position vectors of three points A, B and C with respect to an origin O, are **12a**, **18b** and **10a + 3b** respectively. Express $\overrightarrow{AC}$ and $\overrightarrow{CB}$ in terms of **a** and **b**.
Deduce that A, B and C are collinear and find $AC : CB$.
It is given that $OC = \sqrt{139}$. Show that $A\hat{O}B = \frac{\pi}{3}$.

(*b*) Let $ABCD$ be a rhombus with $AB = 2$ m and $B\hat{A}D = \frac{\pi}{3}$. Forces of magnitude 10 N, 2 N, 6 N, P N and Q N act along AD, BA, BD, DC and CB respectively, in the directions indicated by the order of the letters. It is given that the resultant force is of magnitude 10 N and its direction is in the direction parallel to BC in the sense from B to C. Find the values of P and Q. Also, find the distance from A to the point where the line of action of the resultant force meets BA produced.



Now, a couple of moment M Nm acting in the counterclockwise sense and two forces, each of magnitude F N acting along CB and DC in the directions indicated by the order of the letters, are added to the system so that the resultant force passes through the points A and C. Find the values of F and M.

(*a*) $\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC}$

$= \overrightarrow{OC} - \overrightarrow{OA}$ (5)

$= 10\mathbf{a} + 3\mathbf{b} - 12\mathbf{a}$

$= -2\mathbf{a} + 3\mathbf{b}$ (5)

$\overrightarrow{CB} = \overrightarrow{OB} - \overrightarrow{OC}$ (5)

$= 18\mathbf{b} - (10\mathbf{a} + 3\mathbf{b}) = -10\mathbf{a} + 15\mathbf{b}$ (5)

20

$\overrightarrow{CB} = 5\overrightarrow{AC}$ (5)

$\therefore A, B$ and C are collinear (5)

and $AC : CB = 1 : 5$ (5)

15

$OC = \sqrt{139} \Rightarrow \overrightarrow{OC} \cdot \overrightarrow{OC} = 139$ (5)

$(10\mathbf{a} + 3\mathbf{b}) \cdot (10\mathbf{a} + 3\mathbf{b}) = 139$ (5)

$100|\mathbf{a}|^2 + 60\mathbf{a} \cdot \mathbf{b} + 9|\mathbf{b}|^2 = 139$ (5)

$60\mathbf{a} \cdot \mathbf{b} = 30$

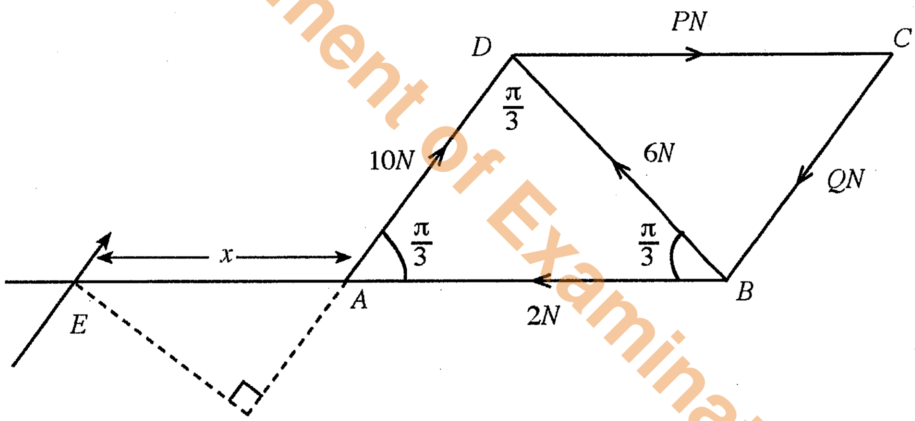
$\mathbf{a} \cdot \mathbf{b} = \frac{1}{2}$ (5)

$|\mathbf{a}|\ |\mathbf{b}|\ \cos A\hat{O}B = \frac{1}{2}$ (5)

$\therefore A\hat{O}B = \frac{\pi}{3}$ (5)

30

(*b*)



(10)

$R = 10N$ (from B to C)

$\uparrow \quad 10 \sin \frac{\pi}{3} = 10 \sin \frac{\pi}{3} - Q \sin \frac{\pi}{3} - 6 \sin \frac{\pi}{3}$ (10)

$\therefore Q = 6$ (5)

$\rightarrow \quad 10 \cos \frac{\pi}{3} = P - 2 - 6 \cos \frac{\pi}{3} - 6 \cos \frac{\pi}{3} + 10 \cos \frac{\pi}{3}$ (10)

$\therefore P = 8$ (5)

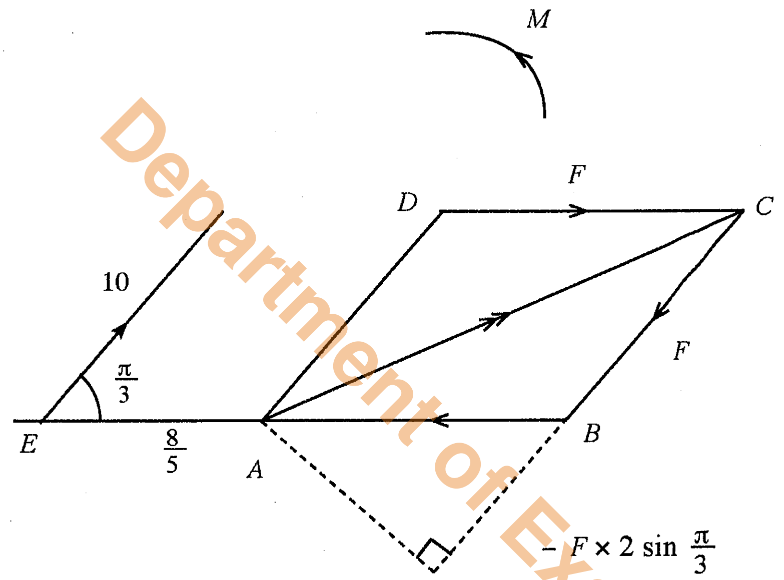
40

$E \curvearrowright \quad 10x \sin\frac{\pi}{3} - 6x\,(2+x)\sin\frac{\pi}{3} - 8x2\sin\frac{\pi}{3} + 6\,(2+x)\sin\frac{\pi}{3} = 0$ (10)

$10x\,\frac{\sqrt{3}}{2} = 8\sqrt{3}$

$x = \frac{8}{5}\, m$ (5)

15



$A \curvearrowleft \quad -10 \times \frac{8}{5}\sin\frac{\pi}{3} + M - F \times 2\sin\frac{\pi}{3} = 0$ (10)

$M = F \times 2\sqrt{3} + 8\sqrt{3}$ (5)

$C \curvearrowleft \quad M - 10\,(2 + \frac{8}{5})\sin\frac{\pi}{3} = 0$ (5)

$M = 10 \times \frac{18}{5} \times \frac{\sqrt{3}}{2}$

$= 18\sqrt{3}$ (5)

$F = \frac{18\sqrt{3} - 8\sqrt{3}}{2\sqrt{3}} = 5.$ (5)

30

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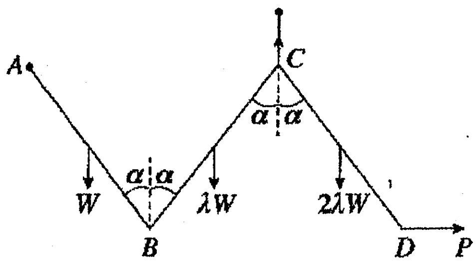
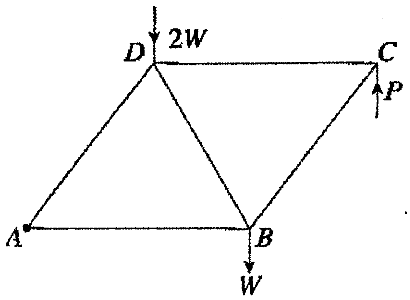
15. (*a*) Three uniform rods AB, BC and CD, each of length $2a$ are smoothly joined at the ends B and C. The weights of the rods AB, BC and CD are W, λW and $2\lambda W$, respectively. The end A is smoothly hinged to a fixed point. The rods are kept in equilibrium in a vertical plane by a light inextensible string attached to the joint C and to a fixed point vertically above C and by a horizontal force P applied to the end D such that A and C are at the same horizontal level and each of the rods making an angle α with the vertical, as shown in the figure. Show that $\lambda = \frac{1}{3}$.



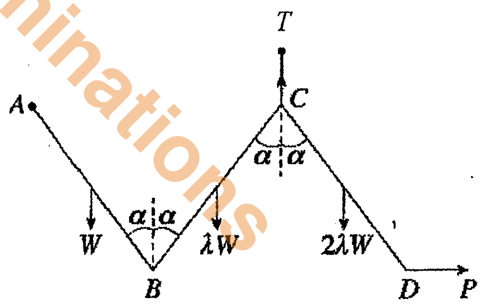
Show also that the horizontal and vertical components of the force exerted on AB by CB at B are $\frac{W}{3}\tan\alpha$ and $\frac{W}{6}$, respectively.

(*b*) The framework shown in the adjoining figure is made from light rods AB, BC, CD, DA and BD, each of length $2a$, freely jointed at A, B, C and D. There are loads of W and $2W$ at B and D, respectively. The framework is smoothly hinged at A to a fixed point and kept in equilibrium with AB horizontal by a vertical force P applied to it at C, as shown in the figure. Find the value of P in terms of W.



Draw a stress diagram using Bow's notation and hence, find the stresses in the rods stating whether they are tensions or thrusts.

(*a*)



Taking moments :

about C for CD

$C\curvearrowright$ $2\lambda Wa\sin\alpha - P\,2a\cos\alpha = 0$ (5)

$\therefore\; P = \lambda W\tan\alpha$ (5)

about B for BC and CD

$C\curvearrowright$ $\lambda Wa\sin\alpha - T\,2a\sin\alpha + 2\lambda W\,3a\sin\alpha = 0$ (10)

$\therefore\; T = \frac{7}{2}\lambda W$ (5)

about A for AB, BC and CD

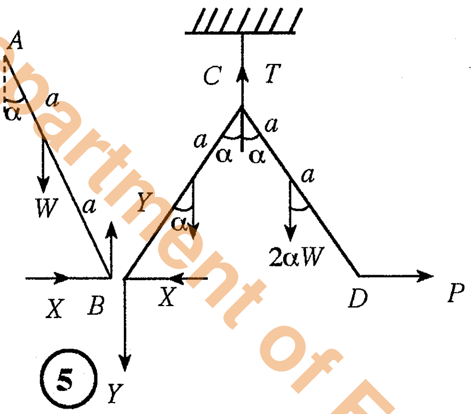
$$A \curvearrowright \quad Wa\sin\alpha + \lambda W\,3a\sin\alpha - T\,4a\sin\alpha + 2\lambda W\,5a\sin\alpha - P\,2a\cos\alpha = 0$$ (10)

$$W\sin\alpha + 13\lambda W\sin\alpha - 14\lambda W\sin\alpha - \lambda W\tan\alpha\;2\cos\alpha = 0$$ (5)

$$1 - \lambda - 2\lambda = 0$$

$$\Rightarrow \lambda = \frac{1}{3}$$ (5)

45



(5)

For BC and CD

$$\uparrow \quad Y + 3\lambda W - T = 0$$

$$\therefore Y = \frac{7}{2}\lambda W - 3\lambda W$$ (5)

$$= \frac{\lambda W}{2}$$

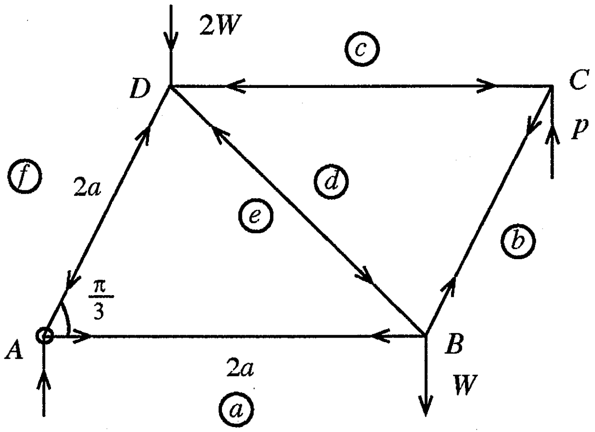
$$= \frac{W}{6}$$

$$\leftarrow \quad X - P = 0$$

$$\therefore X = \frac{1}{3}W\tan\alpha$$ (5)

15

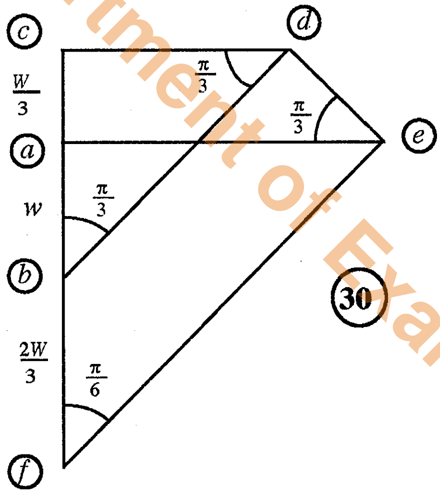
(b)



A ↻ $2Wa + W2a - P3a = 0$

$P = \frac{4W}{3}$ (10)

10



(10 for each joints)

(30)

30

Rod	Tension	Thrust
AB	$\frac{5\sqrt{3}\,W}{9}$	–
BC	$\frac{8\sqrt{3}\,W}{9}$	–
CD	–	$\frac{4\sqrt{3}\,W}{9}$
DA	–	$\frac{10\sqrt{3}\,W}{9}$
BD	–	$\frac{2\sqrt{3}\,W}{9}$

(10) (10) (10) (10) (10)

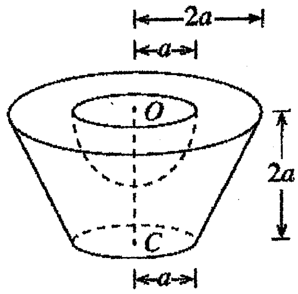
50

16. Show that the centre of mass of

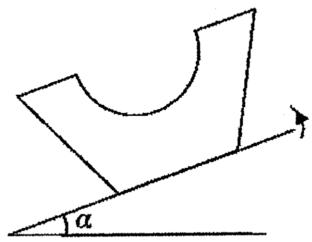
(i) a uniform solid right circular cone of base radius r and height h is at a distance $\frac{h}{4}$ from the centre of the base,

(ii) a uniform solid hemisphere of radius r is at a distance $\frac{3r}{8}$ from its centre.

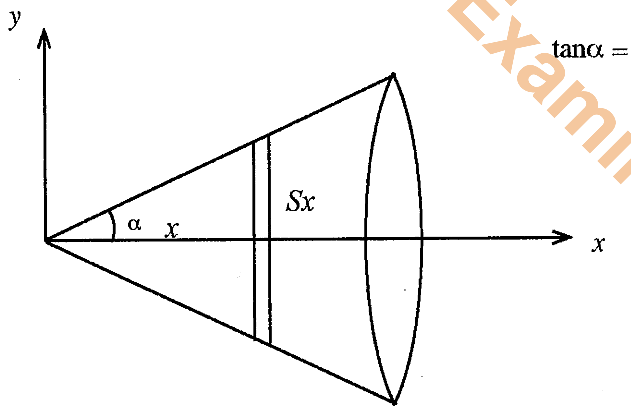
The adjoining figure shows a mortar S made by removing a solid hemisphere from a frustum of a solid uniform right circular cone having base radius $2a$ and height $4a$. The radius and the centre of the upper circular face of the frustum are $2a$ and O, respectively, and those for the lower circular face are a and C, respectively. The height of the frustum is $2a$. The radius and the centre of the removed solid hemisphere are a and O, respectively. Show that the centre of mass of mortar S lies at a distance $\frac{41}{48}a$ from O.



Mortar S is placed on a rough horizontal plane with its lower circular face touching the plane. Now, the plane is tilted upwards slowly. The coefficient of friction between the mortar and the plane is 0.9. Show that if $\alpha < \tan^{-1}(0.9)$, then the mortar stays in equilibrium, where α is the inclination of the plane to the horizontal.



(i) Uniform solid right circular cone



$\tan\alpha = \frac{r}{h}$

By symmetry, the centre of mass lies on the x - axis. (5)

$Sx = \pi (x\tan\alpha)^2\, Sx\, \rho$, where ρ is the density.

$$\bar{x} = \frac{\int_0^h \pi\tan^2\alpha\, \rho x^2 . x\, dx}{\int_0^h \pi\tan^2\alpha\, \rho x^2\, dx}$$ (5) (5)

$$= \frac{\left.\frac{x^4}{4}\right|_0^h}{\left.\frac{x^3}{3}\right|_0^h}$$ (5) (5)

$$= \frac{\frac{h^4}{4}}{\frac{h^3}{3}} = \frac{3h}{4}.$$

$\therefore$ The distance from the centre of the base $= h - \frac{3h}{4}$

$= \frac{h}{4}$ (5)

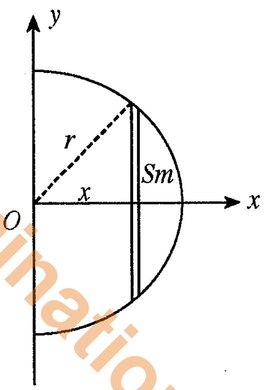
30

(i) Uniform solid hemisphere

By symmetry, the centre of mass lies on the x - axis. (5)

$Sm = \pi (r^2 - x^2)\, \delta x\, \sigma,$

where σ is the density

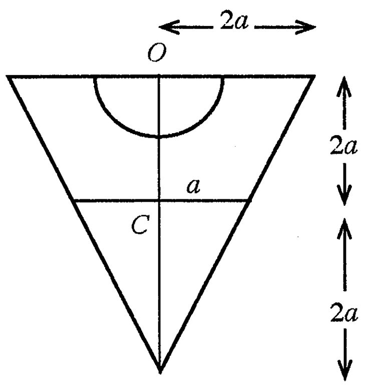


$$\bar{x} = \frac{\int_0^r \pi (r^2 - x^2)\, \sigma x\, dx}{\int_0^r \pi (r^2 - x^2)\, \sigma\, dx}$$ (5) (5)

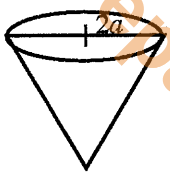
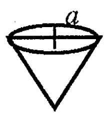
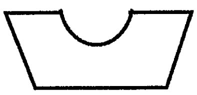
$$= \frac{\left.\left(\frac{r^2 x^2}{2} - \frac{x^4}{4}\right)\right|_0^r}{\left.\left(r^2 x - \frac{x^3}{3}\right)\right|_0^r}$$ (5) (5)

$$= \frac{\frac{r^4}{2} - \frac{r^4}{4}}{r^3 - \frac{r^3}{3}}$$

$$= \frac{3r}{8}$$ (5)



density σ

Object	Mass	Distance from *O*
	$\frac{16}{3}\pi a^3\rho$ (5)	a (5)
	$\frac{2}{3}\pi a^3\rho$ (5)	$\frac{5a}{2}$ (5)
	$\frac{2}{3}\pi a^3\rho$ (5)	$\frac{3a}{8}$ (5)
	$4\pi a^3\rho$ (5)	$\bar{x}$

By symmetry, the centre of mass lies on the axis of symmetry. (5)

$$4\pi a^3\rho\,\bar{x} = \frac{16}{3}\pi a^3\rho a - \frac{2}{3}\pi a^3\rho\,\frac{5a}{2} - \frac{2}{3}\pi a^3\rho a\,\frac{3a}{8}$$ (20)

$$4\bar{x} = \frac{16}{3}a - \frac{5a}{2} - \frac{a}{4}$$ —

$$\bar{x} = \frac{41a}{48}$$ (5)

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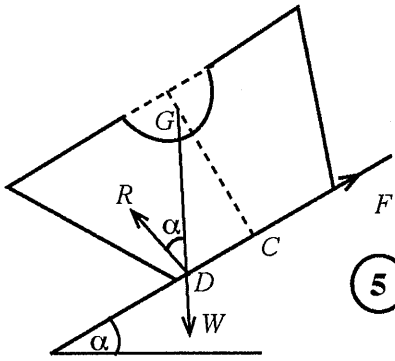

(5)

To prevent sliding	To prevent rolling
$\mu \geq \tan\alpha$ and so	$CD < a$ and so
$0.9 \geq \tan\alpha$ (10)	$CG \tan\alpha < a.$
i.e. $\alpha \leq \tan^{-1}(0.9)$	i.e. $\frac{55a}{48}\tan\alpha < a$ (10)
	and so $\alpha < \tan^{-1}\left(\frac{48}{55}\right)$

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17. (*a*) In a certain factory, machine A makes 50% of the items and the rest are made by machines B and C. It is known that 1%, 3% and 2% of the items made by A, B and C respectively are defective. The probability that a randomly selected item is defective is given to be 0.018. Find the percentages of items made by the machines B and C.

Given that a randomly selected item is defective, find the probability that it was made by the machine A.

(*b*) The time taken (in minutes) to travel to work from their homes of 100 employees of a certain factory are given in the following table:

Time taken	Number of employees
0 – 20	10
20 – 40	30
40 – 60	40
60 – 80	10
80 – 100	10

Estimate the mean, standard deviation and the mode of the distribution given above.

Later, all of the employees in the class interval 80 – 100 moved closer to the factory. It has changed the frequency of the class interval 80 – 100 from 10 to 0 and the frequency of the class interval 0 – 20 from 10 to 20.

Estimate the mean, standard deviation and the mode of the new distribution.

(*a*)

	A	B	C
Probability of Production	$\frac{1}{2}$	p	$\frac{1}{2} - p$
Probability of defects	$\frac{1}{100}$	$\frac{3}{100}$	$\frac{2}{100}$

D – randomly selected item is defective

$$P(D) = P(D/A)\,P(A) + P(D/B)\,P(B) + P(D/C)\,P(C)$$

$$0.018 = \frac{1}{100} \times \frac{1}{2} + \frac{3}{100} \times p + \frac{2}{100} \times \left(\frac{1}{2} - p\right) \quad (10)$$

$$3.6 = 1 + 6p + 2 - 4p$$

$$\Rightarrow p = 0.3 \quad (5)$$

$\therefore$ The percentage of items made by: machine B is 30% (5)

and machine C is 20% (5) **25**

$$P(A/D) = \frac{P(D/A)\ P(A)}{P(D)}$$ (10)

$$= \frac{\frac{1}{100} \times \frac{1}{2}}{0.018}$$ (10)

$$= \frac{1}{100 \times 2}$$

$$= \frac{1}{\frac{18}{1000}}$$

$$= \frac{5}{18}$$ (5)

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Time taken	f (5)	Mid Point x	$y = \frac{1}{10}x$	y^2	fy (5)	fy^2 (5)
0 - 20	10	10	1	1	10	10
20 - 40	30	30	3	9	90	270
40 - 60	40	50	5	25	200	1000
60 - 80	10	70	7	49	70	490
80 - 100	10	90	9	81	90	810
	100				$\sum fy = 460$ (5)	$\sum fy^2 = 2580$ (5)

$$\mu_y = \frac{\sum fy}{\sum f} = \frac{460}{100} = \frac{23}{5}$$ (5)

and $\sigma_y^2 = \frac{\sum fy^2}{\sum f} - \mu_y^2$

$$= \frac{2580}{100} - \left(\frac{23}{5}\right)^2$$

$$= \frac{116}{25}$$ (5)

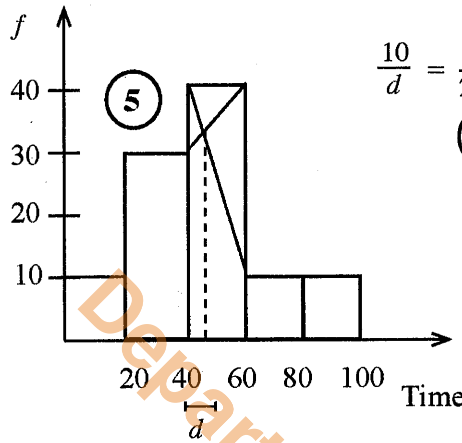
$$\therefore \quad \sigma_y = \sqrt{\frac{116}{25}}$$ (5)

$$= \frac{2\sqrt{29}}{5}$$

$\therefore$ Mean $\mu_x = 10\,\mu_y = 10 \times \dfrac{23}{5} = 46$ (5)

$\therefore$ Standard deviation $\sigma_x = 10\sigma_y = 10 \times \dfrac{2\sqrt{29}}{5} = 4\sqrt{29} \approx 21.54$ (5)

Mode



$\dfrac{10}{d} = \dfrac{30}{20 - d} \Rightarrow d = 5$ (5) $\therefore$ Mode $= 40 + d = 45$ (5)

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(b) For the new distribution:

$$\mu_y = \frac{1}{100}\left[\sum_{1}^{5} fy - f_1 y_1 - f_5 y_5 + 20 \times 1\right]$$

$$= \frac{1}{100}[460 - 10 - 90 + 20] = \frac{380}{100}$$ (5)

$$= \frac{19}{5}$$

$\therefore$ New mean $= 10 \times \dfrac{19}{5} = 38$ (5)

$$\sigma_y^2 = \left[\sum_{1}^{5} fy^2 - f_1 y_1^2 - f_5 y_5^2 + 20 \times 1^2\right] - \left(\frac{19}{5}\right)^2$$

$$= \frac{1}{100}[2580 - 10 - 810 + 20] - \frac{361}{25}$$ (5)

$$= \frac{1780}{100} - \frac{361}{25}$$

$$= \frac{84}{25}$$

$$\therefore \quad \sigma_y = \frac{\sqrt{84}}{5} = \frac{2\sqrt{21}}{5} \quad (5)$$

$\therefore$ New Standard deviation $= 10 \times \frac{2\sqrt{21}}{5} = 4\sqrt{21} \approx 18.33$ (5)

Mode does not change (10) ($\because$ there is no change of the frequencies of the neighbourhood of the mode class)

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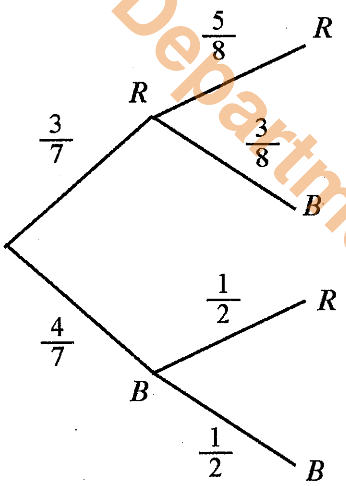
Old Syllabus

8. A bag A contains 3 red balls and 4 black balls, and another bag B contains 4 red balls and 3 black balls. The balls in bag A and bag B are identical in all aspects except for their colour. A ball is drawn at random from bag A and put into bag B. Now, a ball is drawn at random from bag B. Find the probability that

(i) the ball drawn from bag B is black,

(ii) the ball drawn from bag B is black, given that the ball drawn from bag A is red.

A	B
3 Red, 4 Black	4 Red, 3 Black



(i) $P(\text{Ball from } B \text{ is black}) = \frac{3}{7} \times \frac{3}{8} + \frac{4}{7} \times \frac{1}{2} = \frac{9}{56} + \frac{16}{56} = \frac{25}{56}$ (5) (5) (5)

(ii) $P(\text{Black from } B \mid \text{red from } A) = \dfrac{P(\text{Black from } B \text{ and red from } A)}{P(\text{red from } A)}$

$$= \frac{\frac{3}{7} \times \frac{3}{8}}{\frac{3}{7}}$$

$$= \frac{3}{8}$$ (10)

(Or just from the branch from the tree.)

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10. The mean and the standard deviation of marks obtained by students of a class for a question paper in statistics are 40 and 15, respectively. These marks were transformed using the formula $t = \frac{1}{3}(70 + 2x)$, where x is the original mark. Find the mean and the standard deviation of the transformed marks. The median of the transformed marks is 55. Find the median of the original marks.

$$\mu_t = \frac{1}{3}(70 + 2\mu_0) = \frac{1}{3}(70 + 80) = 50$$ (5) (5)

$$\sigma_t = \frac{2}{3}\sigma_0 = \frac{2}{3} \times 15 = 10$$ (5)

$$M_t = \frac{1}{3}(70 + 2M_0)$$ (5)

$$55 = \frac{1}{3}(70 + 2M_0)$$

$$M_0 = \frac{95}{2} = 47.5$$ (5)

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