# G.C.E. (A.L.) Examination - 2019 10 - Combined Mathematics I (Old Syllabus)

## **Distribution of Marks**

## Paper I

Part A :  $10 \times 25 = 250$ Part B :  $05 \times 150 = 750$ 

## Total = 1000 / 10

Paper I Final Mark = 100

### Common Techniques of Marking Answer Scripts.

It is compulsory to adhere to the following standard method in marking answer scripts and entering marks into the mark sheets.

- 1. Use a red color ball point pen for marking. (Only Chief/Additional Chief Examiner may use a mauve color pen.)
- 2. Note down Examiner's Code Number and initials on the front page of each answer script.
- 3. Write off any numerals written wrong with a clear single line and authenticate the alterations with Examiner's initials.
- 4. Write down marks of each subsection in a  $\triangle$  and write the final marks of each question as a rational number in a with the question number. Use the column assigned for Examiners to write down marks.



#### MCQ answer scripts: (Template)

- 1. Marking templets for G.C.E.(A/L) and GIT examination will be provided by the Department of Examinations itself. Marking examiners bear the responsibility of using correctly prepared and certified templates.
- 2. Then, check the answer scripts carefully. If there are more than one or no answers Marked to a certain question write off the options with a line. Sometimes candidates may have erased an option marked previously and selected another option. In such occasions, if the erasure is not clear write off those options too.
- 3. Place the template on the answer script correctly. Mark the right answers with a 'V' and the wrong answers with a 'X' against the options column. Write down the number of correct answers inside the cage given under each column. Then, add those numbers and write the number of correct answers in the relevant cage.

#### Structured essay type and assay type answer scripts:

- 1. Cross off any pages left blank by candidates. Underline wrong or unsuitable answers. Show areas where marks can be offered with check marks.
- 2. Use the right margin of the overland paper to write down the marks.
- 3. Write down the marks given for each question against the question number in the relevant cage on the front page in two digits. Selection of questions should be in accordance with the instructions given in the question paper. Mark all answers and transfer the marks to the front page, and write off answers with lower marks if extra questions have been answered against instructions.
- 4. Add the total carefully and write in the relevant cage on the front page. Turn pages of answer script and add all the marks given for all answers again. Check whether that total tallies with the total marks written on the front page.

#### Preparation of Mark Sheets.

Except for the subjects with a single question paper, final marks of two papers will not be calculated within the evaluation board this time. Therefore, add separate mark sheets for each of the question paper. Write paper 01 marks in the paper 01 column of the mark sheet and write them in words too. Write paper II Marks in the paper II Column and wright the relevant details. For the subject 51 Art, marks for Papers 01, 02 and 03 should be entered numerically in the mark sheets.

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For n = 1, L.H.S. =  $2 \times 1 - 1 = 1$  and R.H.S. =  $1^2 = 1$  (5)

 $\therefore$  The result is true for n = 1.

Take any  $p \in \mathbb{Z}^+$  and assume that the result is true for n = p.

i.e. 
$$\sum_{r=1}^{p} (2r-1) = p^2$$
. 5  
Now  $\sum_{r=1}^{p+1} (2r-1) = \sum_{r=1}^{p} (2r-1) + (2(p+1)-1)$  5  
 $= p^2 + (2p+1)$   
 $= (p+1)^2$ . 5

Hence, if the result is true for n = p, then it is true for n = p + 1. We have already proved that the result is true for n = 1.

Hence, by the Principle of Mathematical Induction, the result is true for all  $n \in \mathbb{Z}^+$ . (5)

2. Sketch the graphs of y=|4x-3| and y=3-2|x| in the same diagram. Hence or otherwise, find all real values of x satisfying the inequality |2x-3|+|x|<3.



At the point of intersections of the graphs

 $4x - 3 = 3 - 2x \implies x = 1$  (5) -4x + 3 = 3 + 2x  $\implies x = 0$ 

From the graphs, we have,

- $|4x-3| < 3-2 |x| \qquad \Leftrightarrow \quad 0 < x < 1$
- $\therefore |4x-3| + |2x| < 3 \qquad \Leftrightarrow \quad 0 < x < 1$

Replacing x by  $\frac{x}{2}$ , we get  $|2x-3| + |x| < 3 \iff 0 <$ 

$$|x-3| + |x| < 3 \quad \Leftrightarrow \quad 0 < x < 2.$$
 (5)

Hence, the set of all values of *x* satisfying

$$|2x-3| + |x| < 3$$
 is  $\{x : 0 < x < 2\}$ . (5)



3. Sketch, in an Argand diagram, the locus of the points that represent complex numbers z satisfying  $\operatorname{Arg}(z-2-2i) = -\frac{3\pi}{4}$ .

Hence or otherwise, find the minimum value of  $|i\overline{z}+1|$  such that  $\operatorname{Arg}(z-2-2i) = -\frac{3\pi}{4}$ .



Note that

$$|i\overline{z} + 1| = |i(\overline{z} - i)| = |\overline{z} - i| = |\overline{z} + i|$$
$$= |z + i|$$
$$= |z - (-i)|$$
5  
Hence, the minimum of  $|i\overline{z} + 1|$  is equal to PM.

Now, PM = 1 . 
$$\sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}$$
 . (5)

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4. Show that the coefficient of  $x^6$  in the binomial expansion of  $\left(x^3 + \frac{1}{x^2}\right)^7$  is 35. Show also that there **does not exist** a term independent of x in the above binomial expansion.

$$\left(x^{3} + \frac{1}{x^{2}}\right)^{7} = \sum_{r=0}^{7} C_{r} (x^{3})^{r} \left(\frac{1}{x^{2}}\right)^{7-r}$$

$$= \sum_{r=0}^{7} C_{r} x^{5r-14}$$

$$x^{6} : 5r - 14 = 6 \iff r = 4.$$

$$(5)$$

$$\therefore \text{ The coefficient of } x^{6} = C_{4}^{7} = 35$$

For the above expansion to have a term independent of *x*, we must have

5r - 14 = 0. (5)

This is not possible as  $r \in \mathbb{Z}^+$ . 5

5. Show that 
$$\lim_{x \to 3} \frac{\sqrt{x-2}-1}{\sin(\pi(x-3))} = \frac{1}{2\pi}$$
.  
$$\lim_{x \to 3} \frac{\sqrt{x-2}-1}{\sin(\pi(x-3))} = \lim_{x \to 3} \frac{\sqrt{x-2}-1}{\sin(\pi(x-3))} \cdot \frac{(\sqrt{x-2}+1)}{(\sqrt{x-2}+1)}$$
 (5)

$$= \lim_{x \to 3^{+} 0} \frac{x - 3}{\sin(\pi(x - 3))} \cdot \lim_{x \to 3^{+} 3} \frac{1}{(\sqrt{x - 2} + 1)} \quad (5)$$

$$= \lim_{x \to 3^{+} 0} \frac{1}{\frac{\sin(\pi(x - 3))}{\pi(x - 3)}} \cdot \frac{1}{\pi} \cdot \frac{1}{2} \quad (5)$$

$$= 1 \cdot \frac{1}{\pi} \cdot \frac{1}{2}$$

$$(5)$$

$$= \frac{1}{2\pi} \quad (5)$$

6. Show that the area of the region enclosed by the curves  $y = \sin 2x$ ,  $y = -x^2$ ,  $x = \frac{\pi}{2}$  and  $x = \pi$  is  $\left(\frac{7}{24}\pi^3 - 1\right)$ .



Required area

$$= \int_{\frac{\pi}{2}}^{\pi} \left[ (\sin 2x) - (-x)^{2} \right] dx \qquad 10$$

$$= \left[ -\frac{1}{2} \cos 2x + \frac{1}{3} x^{3} \right]_{\frac{\pi}{2}}^{\pi} \qquad 5$$

$$= \left( -\frac{1}{2} + \frac{1}{3} \pi^{3} \right) - \left[ +\frac{1}{2} + \frac{1}{3} \left( \frac{\pi}{2} \right)^{3} \right] \qquad 5$$

$$= \frac{7\pi^{3}}{24} - 1. \qquad 5$$

7. A curve C is given parametrically by  $x = e^t(1+t^2)$  and  $y = e^t(1-t^2)$  for  $t \in \mathbb{R}$ . Show that  $\frac{dy}{dx} = -\frac{(t^2+2t-1)}{(t+1)^2}$  for  $t \neq -1$ . Find the equation of the tangent line to the curve C at the point  $P \equiv (1,1)$  on it.

Curve

$$x = e^{t}(1+t^{2}), \qquad y = e^{t}(1-t^{2}), \ t \in \mathbb{R}$$

$$\frac{dx}{dt} = e^{t}(2t+1+t^{2}), \quad 5 \qquad \frac{dy}{dt} = e^{t}(-2t+1-t^{2}) \quad 5$$

$$\therefore \ \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = -\frac{(t^{2}+2t-1)}{(t+1)^{2}} \quad \text{for } t \neq -1$$

$$5$$

At the point P(1, 1), we have t = 0 and  $\frac{dy}{dx} = 1$ .

The equations of tangent at P is y - 1 = 1 (x - 1).

5

i.e.  $y \equiv x$ .

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8. Let  $l_1$  and  $l_2$  be the straight lines given by x + y = 4 and 4x + 3y = 10, respectively. Two distinct points P and Q are on the line  $l_1$  such that the perpendicular distance from each of these points to the line  $l_2$  is 1 unit. Find the coordinates of P and Q.



Any point on the line  $l_1$  can be written in the form

 $(t, 4 - t), t \in \mathbb{R} \cdot 5$ Let  $P = (t_1, 4 - t_1)$ Perpendicular distance from P to  $l_2 = \left| \frac{4t_1 + 3(4 - t_1) - 10}{\sqrt{4^2 + 3^2}} \right| = 1$   $\therefore |t_1 + 2| = 5$   $\therefore t_1 = -7 \text{ or } t_1 = 3 \quad 5$ The coordinates of P and Q are  $(-7, 11) \text{ and } (3, 1). \quad 5 + 5$ 

9. Show that the point  $A \equiv (-7, 9)$  lies outside the circle  $S \equiv x^2 + y^2 - 4x + 6y - 12 = 0$ . Find the coordinates of the point on the circle S=0 nearest to the point A.

The centre C of S = 0 is (2, -3). **5** The radius R of S = 0 is  $\sqrt{4+9+12} = \sqrt{25} = 5$ . **5**  $CA^2 = 9^2 + 12^2 = 15^2 \Rightarrow CA = 15 > R = 5$ . **5** 

 $\therefore$  Point *A* lies outside the given circle.



The point on the circle S = 0 nearest to point A is the point P at which CA meets S = 0. Note that CP : PA = 5:10= 1:2 5

$$\therefore P = \left(\frac{2 \times 2 + 1 (-7)}{3}, \frac{2 (-3) + 1 \times 9}{3}\right)$$
  
i.e.  $P = (-1, 1)$  5

10. Let 
$$t = \tan \frac{\theta}{2}$$
 for  $\theta \neq (2n+1)\pi$ , where  $n \in \mathbb{Z}$ . Show that  $\cos \theta = \frac{1-t^2}{1+t^2}$ .  
Deduce that  $\tan \frac{\pi}{12} = 2 - \sqrt{3}$ .

$$\cos \theta = \cos^{2} \frac{\theta}{2} - \sin^{2} \frac{\theta}{2} \quad (5)$$

$$= \frac{\cos^{2} \frac{\theta}{2} - \sin^{2} \frac{\theta}{2}}{\cos^{2} \frac{\theta}{2} + \sin^{2} \frac{\theta}{2}} = \frac{1 - \tan^{2} \frac{\theta}{2}}{1 + \tan^{2} \frac{\theta}{2}} \quad \text{for } \theta \neq (2n + 1) \pi.$$

$$= \frac{1 - t^{2}}{1 + t^{2}}$$
Let  $\theta = \frac{\pi}{6}$ . Then  $\sqrt{\frac{3}{2}} = \frac{1 - t^{2}}{1 + t^{2}}$ 

$$(5)$$

$$\Rightarrow \sqrt{3} (1 + t^{2}) = 2 (1 - t^{2})$$

$$(2 + \sqrt{3}) t^{2} = 2 - \sqrt{3}$$

$$\therefore t^{2} = \frac{(2 - \sqrt{3})}{(2 + \sqrt{3})} \quad (5)$$

$$= (2 - \sqrt{3})^{2}$$

$$\Rightarrow t = \tan \frac{\pi}{12} = 2 - \sqrt{3} \quad (\because \tan \frac{\pi}{12} > 0)$$

11. (a) Let p∈R and 0<p≤1. Show that 1 is not a root of the equation p<sup>2</sup>x<sup>2</sup> + 2x + p = 0. Let α and β be the roots of this equation. Show that α and β are both real. Write down α + β and αβ in terms of p, and show that

$$\frac{1}{(\alpha-1)} \cdot \frac{1}{(\beta-1)} = \frac{p^2}{p^2 + p + 2}$$

Show also that the quadratic equation whose roots are  $\frac{\alpha}{\alpha-1}$  and  $\frac{\beta}{\beta-1}$  is given by  $(p^2+p+2)x^2-2(p+1)x+p=0$  and that both of these roots are positive.

(b) Let c and d be two non-zero real numbers and let f(x) = x<sup>3</sup> + 2x<sup>2</sup> - dx + cd. It is given that (x-c) is a factor of f(x) and that the remainder when f(x) is divided by (x-d) is cd. Find the values of c and d.
For these values of c and d, find the remainder when f(x) is divided by (x + 2)<sup>2</sup>.

Suppose that 1 is a root of  $p^2 x^2 + 2x + p = 0$ . *(a)* By substituting x = 1, we must have  $p^2 + 2 + p = 0$ . (5) This is impossible, as p > 0 implies that  $p^2 + 2 + p > 0$ . (5) 10  $\therefore$  1 is not a root of  $p^2 x^2 + 2x + p = 0$ The discriminant  $\Delta = 2^2 - 4p^2$ . p 10  $= 4(1-p^3)$  $\geq 0 (:: 0 (5)$  $\therefore \alpha \text{ and } \beta \text{ are both real.} (5)$ 20  $\alpha + \beta = -\frac{2}{p^2}$  and  $\alpha \beta = \frac{1}{p}$  (5) + (5) Now,  $\frac{1}{(\alpha-1)} \cdot \frac{1}{(\beta-1)} = \frac{1}{(\alpha\beta-(\alpha+\beta)+1)}$ 5 =  $\frac{1}{\frac{1}{p} + \frac{2}{p^2} + 1}$  $= \frac{p^2}{p^2 + p + 2} \cdot \left(\mathbf{5}\right)$ 20

Now

$$\frac{a}{a-1} + \frac{\beta}{\beta-1} = \frac{a(\beta-1)+\beta(a-1)}{(a-1)(\beta-1)}$$

$$= \frac{2a\beta-(a+\beta)}{(a-1)(\beta-1)}$$

$$= \left(\frac{2}{p} + \frac{2}{p^2}\right) \cdot \frac{p^2}{p^2+p+2}$$

$$= \frac{2(p+1)}{p^2} \cdot \frac{p^2}{p^2+p+2}$$

$$= \frac{2(p+1)}{p^2+p+2}$$

$$= \frac{2(p+1)}{p^2+p+2}$$

$$\frac{\beta}{\beta-1} = \frac{a\beta}{(a-1)(\beta-1)}$$

and

$$\frac{\alpha}{\alpha - 1} \cdot \frac{\beta}{\beta - 1} = \frac{\alpha\beta}{(\alpha - 1)(\beta - 1)}$$
$$= \frac{1}{p} \cdot \frac{p^2}{p^2 + p + 2}$$
$$= \frac{p}{p^2 + p + 2} \cdot 5$$

Hence, the required quadratic equation is given by

$$x^{2} - \frac{2(p+1)}{p^{2} + p + 2} x + \frac{p}{p^{2} + p + 2} = 0$$

$$(10)$$

$$\Rightarrow (p^{2} + p + 2) x^{2} - 2 (p+1) x + p = 0$$

$$(5)$$
Moreover, note that  $\frac{a}{(a-1)}$  and  $\frac{\beta}{(\beta-1)}$  are both real,
$$\frac{a}{(a-1)} + \frac{\beta}{(\beta-1)} = \frac{2(p+1)}{p^{2} + p + 2} > 0, \quad (\because p > 0),$$

$$(5)$$
and  $\frac{a}{(a-1)} \cdot \frac{\beta}{(\beta-1)} = \frac{p}{p^{2} + p + 2} > 0, \quad (\because p > 0).$ 
Hence, both of these roots are possitive.
$$(5)$$

Confidential

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(b) 
$$f(x) = x^{3} + 2x^{2} - dx + cd$$
  
Since  $(x - c)$  is a factor,  $f(c) = 0$ . (5)  
 $\Rightarrow c^{3} + 2c^{2} - dc + cd = 0$  (5)  
 $\Rightarrow c^{2} (c + 2) = 0$   
 $\Rightarrow c = -2$  ( $\because c \neq 0$ ) (5)

Since, when f(x) is divided by (x - d), the remainder is *cd*, we have

$$f(d) = cd.$$

Let Ax + B be the remainder, when f(x) is divided by  $(x + 2)^2$ .

Then  $f(x) = (x + 2)^2 Q(x) + (Ax + B)$ , where Q(x) is a polynomial of degree 1.

So,  $x^{3} + 2x^{2} + x + 2 = (x + 2)^{2} Q(x) + Ax + B.$  Substituting x = -2, we obtain 0 = -2A + B. 5

By differentiating, we have

$$3x^{2} + 4x + 1 = (x + 2)^{2} Q'(x) + 2Q(x)(x + 2) + A.$$
 (5)

Again by substituting x = -2, we obtain



25

Aliter

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By long division we have,

$$x^{2} + 4x + 4 \qquad x^{3} + 2x^{2} + x + 2$$

$$x^{3} + 4x^{2} + 4x$$

$$-2x^{2} - 3x + 2$$

$$-2x^{2} - 8x - 8$$

$$5x + 10.$$

$$x^{3} + 2x^{2} + x + 2 = (x^{2} + 4x + 4) (x - 2) + (5x + 10)$$

$$\therefore \text{ Required remainder is } 5x + 10.$$

$$10$$

(ii)

(a) 
$$P_1 = \{A, B, C, D, E, 1, 2, 3, 4\}$$
 and  $P_2 = \{F, G, H, I, J, 5, 6, 7, 8\}$ 

(i) The number of different ways of choosing 3 different letters and 3 different

digits from  $P_1 = {}^5C_3 \cdot {}^4C_3$  (10)

Hence the number of passwords that can be formed by choosing all 6 elements from  $P_1$ 

\_ \_ \_ \_ \_ \_ \_ \_ \_ \_ \_ \_ \_

$$= {}^{5}C_{3} \cdot {}^{4}C_{3} \cdot 6! \quad (5)$$
  
= 28800 (5)

Different ways of selecting					
from P <sub>1</sub>		from P <sub>2</sub>		Number of Passwords	
Letters	Digits	Letters	Digits		]
3	_	_	3	${}^{5}C_{3} \cdot {}^{4}C_{3} \cdot 6! = 28800$	
2	1	1	2	$ \overset{5}{C_2} \cdot \overset{4}{C_1} \cdot \overset{5}{C_1} \cdot \overset{4}{C_2} \cdot 6! = 864000 $	
1	2	2	1	$ \overset{5}{C}_{1} \cdot \overset{4}{C}_{2} \cdot \overset{5}{C}_{2} \cdot \overset{4}{C}_{1} \cdot 6 ! = 864000 $	
_	3	3	_	${}^{4}C_{3} \cdot {}^{5}C_{3} \cdot 6! = 28800$	

Hence, the number of different passwords that can be formed by choosing 3 elements

from 
$$P_1$$
 and the other 3 elements from  $P_2 = 28800 + 864000 + 864000 + 28800 = 1785600$   
(b)  $U_r = \frac{1}{r(r+1)(r+3)(r+4)}$  and  $V_r = \frac{1}{r(r+1)(r+2)}$ ;  $r \in \mathbb{Z}^+$ .  
Then,  
 $V_r = V_{r+2} = \frac{1}{r(r+1)(r+2)} - \frac{1}{(r+2)(r+3)(r+4)}$   
 $= \frac{(r+3)(r+4) - r(r+1)}{r(r+1)(r+2)(r+3)(r+4)}$   
 $= \frac{6(r+2)}{r(r+1)(r+2)(r+3)(r+4)}$   
 $= 6 U_r$  (5)

Now note that,

13. (a) Let 
$$\mathbf{A} = \begin{pmatrix} a & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$
,  $\mathbf{B} = \begin{pmatrix} 2 & 1 & 3 \\ 1 & -a & 4 \end{pmatrix}$  and  $\mathbf{C} = \begin{pmatrix} b & -2 \\ -1 & b+1 \end{pmatrix}$  be matrices such that  
 $\mathbf{AB^{T}} = \mathbf{C}$ , where  $a, b \in \mathbb{R}$ .  
Show that  $a = 2$  and  $b = 1$ .  
Show also that,  $\mathbf{C}^{-1}$  **does not** exist.  
Let  $\mathbf{P} = \frac{1}{2}(\mathbf{C} - 2\mathbf{I})$ . Write down  $\mathbf{P}^{-1}$  and find the matrix  $\mathbf{Q}$  such that  $2\mathbf{P}(\mathbf{Q} + 3\mathbf{I}) = \mathbf{P} - \mathbf{I}$ , where  
I is the identity matrix of order 2.  
(b) Let  $z, z_1, z_2 \in \mathbb{C}$ .  
Show that (i)  $\operatorname{Re} z \le |z|$ , and  
(ii)  $\left|\frac{z_1}{|z_2|}\right| = \frac{|z_1|}{|z_1 + z_2|}$  for  $z_2 \ne 0$ .  
Deduce that  $\operatorname{Re}\left(\frac{z_1}{|z_1 + z_2|}\right) \le \frac{|z_1|}{|z_1 + z_2|}$  for  $z_1 + z_2 \ne 0$ .  
Verify that  $\operatorname{Re}\left(\frac{z_1}{|z_1 + z_2|}\right) + \operatorname{Re}\left(\frac{z_2}{|z_1 + z_2|}\right) = 1$  for  $z_1 + z_2 \ne 0$ ,  
and show that  $|z_1 + z_2| \le |z_1| + |z_2|$  for  $z_1, z_2 \in \mathbb{C}$ .  
(c) In an Argand diagram,  $O$  is the origin and  $OACB$  is a quadrilateral with vertices taken counterclockwise.  
The point  $A$  represents the complex number  $2 + 4\sqrt{3}i$ , and  $A\hat{OC} = \frac{\pi}{3}$ ,  $O\hat{AC} = \frac{\pi}{2}$ ,  $OA = OB$  and

CA = CB. Find the complex numbers represented by the points B and C.

(a) 
$$AB^{T} = \begin{pmatrix} a & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & -a \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 2a - 3 & a - 4 \\ -1 & a \end{pmatrix}$$
  
(10)  
 $AB^{T} = C \iff \begin{pmatrix} 2a - 3 & a - 4 \\ -1 & a \end{pmatrix} = \begin{pmatrix} b & -2 \\ -1 & b + 1 \end{pmatrix}$   
 $\Leftrightarrow 2a - 3 = b, \quad a - 4 = -2 \text{ and } a = b + 1.$  (10)  
 $\Leftrightarrow a = 2 \text{ and } b = 1, \text{ (from any two equations above) and these values satisfy the remaining equation.}$  (30)

10

$$C = \begin{pmatrix} 1 & -2 \\ -1 & 2 \end{pmatrix}$$
$$\begin{vmatrix} 1 & -2 \\ -1 & 2 \end{vmatrix} = 0$$
 5  
$$\therefore C^{-1} \text{ does not exist.} 5$$

For the existence of C<sup>-1</sup> : there must exist  $p, q, r, s \in \mathbb{R}$  such that  $\begin{pmatrix} 1 & -2 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$  (5)  $\Rightarrow p - 2r = 1, -p + 2r = 0, q - 2s = 0 \text{ and } -q + 2s = 1$ This is a contradiction  $\therefore C^{-1}$  does not exist. (5) 10

$$P = \frac{1}{2} (C - 2I) = \frac{1}{2} \left\{ \begin{pmatrix} 1 & -2 \\ -1 & 2 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \right\} = \frac{1}{2} \begin{pmatrix} -1 & -2 \\ -1 & 0 \end{pmatrix}$$
(5)  

$$\Rightarrow P^{-1} = 2 \left( \frac{1}{-2} \right) \begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ -1 & 1 \end{pmatrix}$$
(10)  

$$2P (Q + 3I) = P - I$$
  

$$\Leftrightarrow 2 (Q + 3I) = I - P^{-1}$$
(5)  

$$\therefore 2 (Q + 3I) = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$$
(5)  

$$\Rightarrow Q = \frac{1}{2} \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$$
(5)  

$$= \begin{pmatrix} -\frac{5}{2} & 1 \\ \frac{1}{2} & -3 \end{pmatrix}$$
(5)

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 $(\boldsymbol{b}) \quad \boldsymbol{z}, \ \boldsymbol{z}_1, \ \boldsymbol{z}_2 \in \mathbb{C}.$ 

(i) Let 
$$z = x + iy$$
,  $x, y \in \mathbb{R}$ .  
Re  $z = x \le \sqrt{x^2 + y^2} = |z|$  (5)

(ii) Let 
$$z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$$
 and  $z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$ .  

$$\Rightarrow \frac{z_1}{z_2} = \frac{r_1(\cos \theta_1 + i \sin \theta_1) \times (\cos \theta_2 - i \sin \theta_2)}{r_2(\cos \theta_2 + i \sin \theta_2) \times (\cos \theta_2 - i \sin \theta_2)} = \frac{r_1}{r_2} \frac{\left[\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)\right]}{10}$$

$$\therefore \left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2} = \frac{|z_1|}{|z_2|} \quad (5)$$
Re $\left(\frac{z_1}{z_1 + z_2}\right) \leq \left|\frac{z_1}{|z_1 + z_2|}\right| = \frac{|z_1|}{|z_1 + z_2|} \quad ; \text{ for } z_1 + z_2 \neq 0.$ 
(5) by (i) (5) by (ii) 10

# For $z_1 + z_2 \neq 0$ , we have

$$\frac{z_1}{z_1 + z_2} + \frac{z_2}{z_1 + z_2} = 1$$

$$Re\left(\frac{z_1}{z_1 + z_2} + \frac{z_2}{z_1 + z_2}\right) = 1$$

$$Re\left(\frac{z_1}{z_1 + z_2}\right) + Re\left(\frac{z_2}{z_1 + z_2}\right) = 1$$
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$$\Rightarrow 1 = \operatorname{Re}\left(\frac{z_{1}}{z_{1}+z_{2}}\right) + \operatorname{Re}\left(\frac{z_{2}}{z_{1}+z_{2}}\right) \leq \left|\frac{z_{1}}{z_{1}+z_{2}}\right| + \left|\frac{z_{2}}{z_{1}+z_{2}}\right| \text{ by (i) } 5$$
$$= \frac{|z_{1}|}{|z_{1}+z_{2}|} + \frac{|z_{2}|}{|z_{1}+z_{2}|} \text{ by (ii)}$$
$$= \frac{|z_{1}|+|z_{2}|}{|z_{1}+z_{2}|} 5$$

 $\Rightarrow |z_1 + z_2| \le |z_1| + |z_2| \qquad (\because |z_1 + z_2| > 0)$ 

Now if  $z_1 + z_2 = 0$ , then

 $|z_1 + z_2| = 0 \le |z_1| + |z_2|$ 

Hence, the result is true for all  $z_1, z_2 \in \mathbb{C}$ .

(*c*)



Let the verticles A, B and C represent complex numbers  $z_0, z_1$  and  $z_3$ .

Then 
$$|z_1| = OB = OA = |z_0|$$
, Also  $AOB = 2 \cdot \frac{\pi}{3} \Delta OAC = \Delta OBC$  (5)  
 $\therefore z_1 = z_0 \left(\cos \frac{2\pi}{3} + \sin \frac{2\pi}{3}\right)$  (5)  
 $= (2 + 4\sqrt{3}i) \left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)$   
 $= -7 - i\sqrt{3}$  (5)

Further,  $OC = 2(OA) = 2|z_0|$ .

$$\therefore z_{2} = 2 z_{0} \left( \cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right) \quad (5)$$
$$= 2(2 + 4\sqrt{3}i) \quad \left( \frac{1}{2} + \frac{i\sqrt{3}}{2} \right)$$
$$= -10 + 6\sqrt{3}i \quad (5)$$







(a) For 
$$0 \le \theta \le \frac{\pi}{4}$$
:  
 $x = 2 \sin^2 \theta + \frac{\pi}{3} \Rightarrow dx = 4 \sin \theta \cos \theta \, d\theta$  (5)  
 $x = 3 \Leftrightarrow 2\sin^2 \theta = 0 \Leftrightarrow \theta = 0$  (5)  
 $x = 4 \Leftrightarrow 2\sin^2 \theta = 1 \Leftrightarrow \sin \theta = \frac{1}{\sqrt{2}} \Leftrightarrow \theta = \frac{\pi}{4}$  (5)  
Then  $\int_{3}^{4} \sqrt{\frac{x-3}{5-x}} \, dx = \int_{0}^{\frac{\pi}{4}} \sqrt{\frac{2\sin^2 \theta}{2-2\sin^2 \theta}} \cdot 4 \sin \theta \cos \theta \, d\theta$  (5)  
 $= \int_{0}^{\frac{\pi}{4}} 4 \sin^2 \theta \, d\theta$  (5)  
 $= 2 \left(\theta - \frac{1}{2} \sin 2\theta\right) \left|_{0}^{\frac{\pi}{4}}$  (5)  
 $= \frac{\pi}{2} - 1$  (5)

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(b) 
$$\frac{1}{(x-1)(x-2)} = \frac{A}{(x-1)} + \frac{B}{(x-2)}$$
  
 $\Leftrightarrow 1 = A(x-2) + B(x-1) \text{ for } x \neq 1, 2.$ 

Comparing coefficients of powers of x:

(c) Using the formula 
$$\int_{a}^{b} f(x) dx = \int_{a}^{b} (a + b - x) dx,$$
  
 $\int_{-\pi}^{\pi} \frac{\cos^{2} x}{1 + e^{x}} dx = \int_{-\pi}^{\pi} \frac{\cos^{2} (-x)}{1 + e^{-x}} dx$  (5)  
 $= \int_{-\pi}^{\pi} \frac{e^{x} \cos^{2} x}{1 + e^{x}} dx$  (5)  
 $\frac{2}{\pi} \int_{-\pi}^{\pi} \frac{\cos^{2} x}{1 + e^{x}} dx = \int_{-\pi}^{\pi} \frac{\cos^{2} x}{1 + e^{-x}} dx + \int_{-\pi}^{\pi} \frac{e^{x} \cos^{2} x}{1 + e^{x}} dx$  (5)  
 $= \int_{-\pi}^{\pi} \frac{(1 + e^{x}) \cos^{2} x}{(1 + e^{x})} dx$   
 $= \int_{-\pi}^{\pi} \cos^{2} x dx$  (5)  
 $= \frac{1}{2} \int_{-\pi}^{\pi} (1 + \cos 2x) dx$  (5)  
 $= \frac{1}{2} \left[ x + \frac{1}{2} \sin 2x \right]_{-\pi}^{\pi}$  (5)  
 $\therefore \int_{-\pi}^{\pi} \frac{\cos^{2} x}{1 + e^{x}} = \frac{\pi}{2}$  (5)

16. Write down the coordinates of the point of intersection A of the straight lines 12x-5y-7=0and y=1.

Let l be the bisector of the acute angle formed by these lines. Find the equation of the straight line l.

Let P be a point on l. Show that the coordinates of P can be written as  $(3\lambda + 1, 2\lambda + 1)$ , where  $\lambda \in \mathbb{R}$ .

Let  $B \equiv (6,0)$ . Show that the equation of the circle with the points B and P as ends of a diameter can be written as  $S + \lambda U = 0$ , where  $S \equiv x^2 + y^2 - 7x - y + 6$  and  $U \equiv -3x - 2y + 18$ .

Deduce that S=0 is the equation of the circle with AB as a diameter.

Show that U=0 is the equation of the straight line through B, perpendicular to l.

Find the coordinates of the fixed point which is distinct from B, and lying on the circles with the equation  $S + \lambda U = 0$  for all  $\lambda \in \mathbb{R}$ .

Find the value of  $\lambda$  such that the circle given by S=0 is orthogonal to the circle given by  $S+\lambda U=0$ .

$12x - 5y - 7 = 0$ and $y = 1 \Rightarrow x = 1$ , $y = 1$		
$\therefore A = (1, 1)$	(10)	10

Equations of the bisectors are given by

$$\frac{12x - 5y - 7}{13} = \pm \frac{(y - 1)}{1}$$

$$\Rightarrow 12x - 5y - 7 = 13 (y - 1) \text{ or } 12x - 5y - 7 = -13 (y - 1)$$

$$\Rightarrow 2x - 3y + 1 = 0 \text{ or } 3x + 2y - 5 = 0$$
(5) + (5)

The angle  $\theta$  between y = 1 and 2x - 3y + 1 = 0, is given by

$$\tan \theta = \left| \frac{\frac{2}{3} - 0}{1 + \frac{2}{3}(0)} \right| = \frac{2}{3} < 1 \quad (5)$$
  
$$\therefore l: 2x - 3y + 1 = 0. \quad (5)$$





#### $\therefore$ The coordinates of *C* is given by

$$u = -3x - 2y + 18 = 0$$
  
and  $l = 2x - 3y + 1 = 0$   
$$\Rightarrow x = 4 \text{ and } y = 3$$
  
$$\therefore C = (4, 3) \cdot 5$$

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The circles ;

S = 0 and  $S + \lambda U = 0$  are orthogonal

$$\Rightarrow 2\left(-\frac{1}{2}(3\lambda+7)\right)\left(-\frac{7}{2}\right)+2\left(-\frac{1}{2}(2\lambda+1)\right)\left(-\frac{1}{2}\right) = 6+18\lambda+6$$

$$5 5 5 5$$

$$\Rightarrow 13\lambda = 26 5$$

$$\Rightarrow \lambda = 2.$$

$$20$$

17. (a) Write down  $\sin(A+B)$  in terms of  $\sin A$ ,  $\cos A$ ,  $\sin B$  and  $\cos B$ , and obtain a similar expression for  $\sin(A-B)$ . **Deduce** that  $2\sin A \cos B = \sin (A+B) + \sin (A-B)$  and  $2\cos A \sin B = \sin (A+B) - \sin (A-B).$ **Hence**, solve  $2\sin 3\theta \cos 2\theta = \sin 7\theta$  for  $0 < \theta < \frac{\pi}{2}$ . (b) In a triangle ABC, the point D lies on AC such that BD = DC and AD = BC. Let  $B\hat{A}C = \alpha$ and  $A\ddot{C}B = \beta$ . Using the Sine Rule for suitable triangles, show that  $2\sin\alpha\cos\beta = \sin(\alpha + 2\beta)$ . If  $\alpha : \beta = 3 : 2$ , using the last result in (a) above, show that  $\alpha = \frac{\pi}{6}$ . (c) Solve  $2\tan^{-1}x + \tan^{-1}(x+1) = \frac{\pi}{2}$ . Hence, show that  $\cos\left(\frac{\pi}{4} - \frac{1}{2}\tan^{-1}\left(\frac{4}{3}\right)\right) = \frac{3}{\sqrt{10}}$ . 5  $\sin (A + B) = \sin A \cos B + \cos A \sin B - -$ **(a)** (1)Now  $\sin(A - B) = \sin(A + (-B))$  (5)  $= \sin A \cos (-B) + \cos A \sin (-B)$ 5  $\therefore \sin (A - B) = \sin A \cos B - \cos A \sin B - (2)$ 15  $(1) + (2) \Rightarrow \sin(A+B) + \sin(A-B) = 2\sin A\cos B, (5)$  $(1) - (2) \implies \sin(A+B) - \sin(A-B) = 2\cos A\sin B.$ 10  $0 < \theta < \frac{\pi}{2}.$  $2\sin 3\theta \cos 2\theta = \sin 7\theta$ ,  $\Leftrightarrow \sin 5\theta + \sin \theta = \sin 7\theta \quad (5)$  $\Leftrightarrow \sin 7\theta - \sin 5\theta - \sin \theta = 0$  $\sin (6\theta + \theta) - \sin (6\theta - \theta) - \sin \theta = 0$ 5  $2\cos 6\theta \sin \theta - \sin \theta = 0$  $\sin\theta (2\cos 6\theta - 1) = 0$ ⇔  $\Leftrightarrow \cos 6\theta = \frac{1}{2} \operatorname{since} 0 < \theta < \frac{\pi}{2}, \sin \theta > 0$ 

$$\Rightarrow 6\theta = 2n\pi \pm \frac{\pi}{3}; n \in \mathbb{Z}. \quad \mathbf{5} + \mathbf{5}$$

$$\Rightarrow \theta = \frac{n\pi}{3} \pm \frac{\pi}{18}; n \in \mathbb{Z}.$$

$$\Rightarrow \theta = \frac{\pi}{18}, \frac{5\pi}{18}, \frac{7\pi}{18}, \left(\because 0 < \theta < \frac{\pi}{2}\right) \quad \mathbf{5}$$

$$\mathbf{30}$$



Note that

$$\stackrel{\wedge}{CBD} = \beta, \ \stackrel{\wedge}{ADB} = 2\beta,$$
  
and  $\stackrel{\wedge}{ABD} = \pi - (\alpha + 2\beta)$ 

Using the sine Rule :

for the triangle ABD, we have

$$\frac{BD}{\sin BAD} = \frac{AD}{\sin ABD} \quad (10)$$

$$\Rightarrow \quad \frac{BD}{\sin \alpha} = \frac{AD}{\sin (\pi - (\alpha + 2\beta))}$$

$$\Rightarrow \quad \frac{BD}{\sin \alpha} = \frac{AD}{\sin (\alpha + 2\beta)} \quad (1)$$

for the triangle *BDC*, we have

$$\frac{CD}{\sin DBC} = \frac{BC}{\sin BDC} \quad \textbf{10}$$

$$\Rightarrow \quad \frac{CD}{\sin \beta} = \frac{BC}{\sin 2\beta} \quad \textbf{(2)}$$

 $\therefore BD = DC$  and AD = BC, from (1) and (2), we get

$$\frac{\sin \alpha}{\sin \beta} = \frac{\sin (\alpha + 2\beta)}{\sin 2\beta} \qquad (5)$$

(*c*)

$$\Rightarrow 2 \sin \alpha \cos \beta = \sin (\alpha + 2\beta).$$
(5)
(40)
If  $\alpha : \beta = 3 : 2$ , then we have
$$2 \sin \alpha \cos \frac{2\alpha}{3} = \sin \frac{7\alpha}{3} \quad (5)$$

$$\Rightarrow 2 \sin 3(\frac{\alpha}{3}) \cos 2(\frac{\alpha}{3}) = \sin 7(\frac{\alpha}{3}) \quad (5)$$

$$\Rightarrow \frac{\alpha}{3} = \frac{\pi}{18}, \frac{5\pi}{18}, \frac{7\pi}{18}.$$

$$\Rightarrow \alpha = \frac{\pi}{6}, \frac{15\pi}{18}, \frac{21\pi}{18} \quad (5)$$

$$\because BC = AD < AC, \ \alpha \text{ must be an acute angle.}$$

$$\therefore \alpha = \frac{\pi}{6} \cdot (5)$$
(20)
$$2 \tan^{-1}x + \tan^{-1}(x+1) = \frac{\pi}{2}$$
Let  $\alpha = \tan^{-1}(x)$  and  $\beta = \tan^{-1}(x+1)$ . Note that  $x \neq \pm 1$ .  
Then  $2\alpha + \beta = \frac{\pi}{2} \cdot (5)$ 

$$\Rightarrow 2\alpha = \frac{\pi}{2} - \beta$$

$$\Rightarrow \tan 2\alpha = \tan(\frac{\pi}{2} - \beta) \quad (5)$$

$$\Rightarrow \frac{2\tan \alpha}{1 - \tan^{2}\alpha} = \cot \beta \quad (5) + (5)$$

$$\Rightarrow 2x = 1 - x \quad (\because x \neq \pm 1)$$

$$\Rightarrow x = \frac{1}{3} \cdot (5)$$
(40)

Note that

$$2 \tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{4}{3}\right) = \frac{\pi}{2}.$$

$$\Rightarrow \frac{\pi}{4} - \frac{1}{2} \tan^{-1}\left(\frac{4}{3}\right) = \tan^{-1}\left(\frac{1}{3}\right)$$

$$\Rightarrow \cos\left(\left(\frac{\pi}{4}\right) - \frac{1}{2} \tan^{-1}\left(\frac{4}{3}\right)\right) = \cos\left(\tan^{-1}\left(\frac{1}{3}\right)\right)$$

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